Center for Academic Resources in Engineering (CARE) Peer Exam Review Session

Math 220 — Calculus I

Midterm 1 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Feb. 16th, 4:00pm - 5:30pm - Geo, Lucy, Noah

Session 2: Feb. 18nd, 6:00pm - 7:30pm -Sophia, Jiya, Emma

Can't make it to a session? Here's our schedule by course:

https://care.grainger.illinois.edu/tutoring/schedule-by-subject

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/844
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. Let $f(x) = 8x^2 + 5$. Use the definition of a derivative as a limit to prove that $f'(x) = 16x$. Show each step in your calculation and be sure to use proper terminology in each step of your proof.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{[8(x+h)^2 + 5] - [8x^2 + 5]}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{[8(x^2 + 2xh + h^2) + 5] - [8x^2 + 5]}{h} = \lim_{h\to 0} \frac{8x^2 + 16xh + 8h^2 + 5 - 8x^2 - 5}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{h(16x + 8h)}{h} = \lim_{h\to 0}(16x + 8h) = 16x$$

2. Compute the following limit:

$$\lim_{x\to\infty} \frac{91\sqrt[8]{x} + 3}{5 - 7\sqrt[8]{x}}$$

$$\lim_{x\to\infty} \frac{\sqrt[8]{x}(91 + \frac{3}{\sqrt[8]{x}})}{\sqrt[8]{x}\left(\frac{5}{\sqrt[8]{x}} - 7\right)}$$

$$\lim_{x\to\infty} \frac{91 + \frac{3}{\sqrt[8]{x}}}{\frac{5}{\sqrt[8]{x}} - 7}$$

$$\frac{91 + 0}{0 - 7} = \boxed{-13}$$

3. Write an equation for each horizontal asymptote on the graph of the following function. Use limits to justify your answer. We will learn L'Hopital's Rule and other shortcuts for obtaining limits later. For now, you are not allowed to use these approaches.

$$\frac{56e^{-5x} - 30}{7e^{-5x} + 10}$$

$$\lim_{x\to\infty} \frac{56e^{-5x} - 30}{7e^{-5x} + 10} = \frac{56 * 0 - 30}{7 * 0 + 10} = -3$$

$$\lim_{x\to -\infty} \frac{56e^{-5x} - 30}{7e^{-5x} + 10} = \frac{56}{7} = 8$$

So the equations of the asymptotes are $\boxed{\text{y = -3}}$ and $\boxed{\text{y = 8}}$. A helpful hint for this problem is to think about the graph of $y = e^{-5x}$.

4. Compute the following limits:

(a)

$$\lim_{x\to 0} \frac{19x - 5sin(x)}{2x}$$

(b)

$$\lim_{x\to 0} \frac{e^{6x} - 1}{e^{3x} - 1}$$

(a)

$$lim_{x\to 0}(\frac{19x}{2x} - \frac{5\sin(x)}{2x})$$

$$\lim_{x\to 0}(\frac{19}{2} - \frac{5}{2}\frac{\sin(x)}{x}) = \frac{19}{2} - \frac{5}{2} = \boxed{7}$$

* Remember that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$

(b)

$$\lim_{x\to 0} \frac{(e^{3x})^2 - 1}{e^{3x} - 1} = \lim_{x\to 0} \frac{(e^{3x} - 1)(e^{3x} + 1)}{e^{3x} - 1}$$

$$\lim_{x\to 0} e^{3x} + 1 = e^{3*0} + 1 = 1 + 1 = \boxed{2}$$

5. Determine whether the following statements are always true.

(a) A function which is continuous at point (a) must also be differentiable at (a).

(b) If $f(x) = sin(x^3)$ and $g(x)$ is an odd function, then the composite function $g(f(x))$ is an odd function.

(c) The function $y = \frac{9x-63}{x^2+6x-91}$ has a vertical asymptote at $x = 7$.

(d) If the point $(\frac{1}{4}, -4)$ is on the graph of a one-to-one function $f(x)$, then the point $(4, -\frac{1}{4})$ must also be on the graph of $f^{-1}(x)$.

(a) This statement is **false**. A function which is differentiable at (a) must be continuous at (a), but not the other way around. For example, $F(x) = |x - 2|$ is continuous but not differentiable at 2.

(b) This statement is **true**.

$L = w(x) = (g * f)(x) = w(-x) = g(f(-x)) = g(\sin((-x)^3)) = g(\sin(-x^3))$

$= g(-\sin(x^3)) = -g(\sin(x^3)) = -g(f(x)) = -w(x)$ because sine is odd and since g is odd.

(c) This statement is **false**.

$lim_{x\to 7}\frac{9x-63}{x^2+6x-91} = \lim_{x\to 7}\frac{9(x-7)}{(x-7)(x+13)}$

$\lim_{x\to 7}\frac{9}{x+13} = \frac{9}{20}$

Clearly this does not go to +/- ∞ so there is no asymptote.

(d) This statement is **false**.

($\frac{1}{4}$, -4) is on the graph of $F(x)$ and (-4, $\frac{1}{4}$) is on the graph of $F'(x)$ but not (4, $\frac{-1}{4}$)

6. Determine the x-intercept on the graph of the following function. Simplify your answer.

$$f(x) = e^{9x} - 121e^{7x}$$

To find x-intercept, set $f(x) = 0$.

$$f(x) = 0 \to e^{9x} - 121e^{7x} = 0$$

$$e^{9x} = 121e^{7x}$$

$$\frac{e^{9x}}{e^{7x}} = 121$$

$$e^{2x} = 121$$

Applying ln on both sides of the equations...

$$\ln\left(e^{2x}\right) = \ln(121)$$

$$2x = \ln(121) \to 2x = \ln\left(11^2\right)$$

$$2x = 2\ln(11)$$

$x = \boxed{\ln(11)}$ which gives us what the x-intercept is

7. Evaluate the following limits and write your answers in simplified form.

(a)

$$\lim_{x\to\sqrt{2}}\frac{120\arcsin\left(\frac{x}{2}\right)}{x^2+4}$$

(b)

$$\lim_{x\to\infty} \frac{13 + 5\sin(9e^{3x} + 6)}{x^{10}}$$

(a) First, we can directly plug in the value of the limit:

$$\frac{120\arcsin\left(\frac{\sqrt{2}}{2}\right)}{(\sqrt{2})^2 + 4}$$

$$\frac{120(\frac{\pi}{4})}{2+4}$$

$$\frac{120(\frac{\pi}{4})}{6}$$

$$20(\frac{\pi}{4}) = \boxed{5\pi}$$

(b) Within this specific function, the range of sine fluctuates between -1 and 1:
$-1 < \sin(9e^{3x} + 6) < 1$: Now multiply by 5

$-5 < 5\sin(9e^{3x} + 6) < 5$: Add 13 to all sides

$8 < 13 + 5\sin(9e^{3x} + 6) < 18$: Divide all sides by x^{10}

$$\frac{8}{x^{10}} < \frac{13 + 5\sin(9e^{3x} + 6)}{x^{10}} < \frac{18}{x^{10}}$$

$$\lim_{x\to\infty} \frac{8}{x^{10}} = 0 \text{ and } \lim_{x\to\infty} \frac{18}{x^{10}} = 0$$

So, by the squeeze theorem, it must be true that:

$$\lim_{x\to\infty} \frac{13 + 5sin(9e^{3x} + 6)}{x^{10}} = \boxed{0}$$

8. State the domain of the following function using interval notation.

$$f(x) = \frac{8\sin(x^2 - 25)}{4 - \sqrt{2x - 6}}$$

Notice the following:

- $8\sin(x^2 - 25)$ has no restriction on the domain
- From $\sqrt{2x-6}$, we know $2x - 6 \geq 0 \implies x \geq 3$

- The denominator is 0 when $4-\sqrt{2x-6}=0$, so $x \neq 11$ since we need a non-zero denominator.

Thus, $f(x)$ is valid on a domain from 3 (inclusive) to infinity, skipping 11. This is written as

$$\boxed{x \in [3, 11) \cup (11, \infty)}$$