Center for Academic Resources in Engineering (CARE)
Peer Exam Review Session

Math 241 – Calculus III

Midterm 4 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Saturday - Dec. 7, 4:00-6:00 pm in 3025 CIF, Cami, Gabe

Session 2: Sunday - Dec 8, 5:00-7:00 pm in 3025 CIF, Pallab, Rose, Kewal

Can't make it to a session? Here's our schedule by course:

https://care.grainger.illinois.edu/tutoring/schedule-by-subject

Solutions will be available on our website after the last review session that we host.

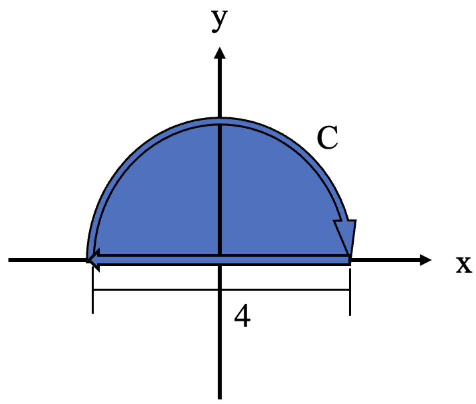
Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/845
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. Evaluate $\int_C F \cdot dr$ where $F(x,y) = \langle 3y^2 - \cos y, x\sin y\rangle$ and C is a counterclockwise path shown below.



Green's Theorem states that $\int_C F \cdot dr = \oint_{-C} Pdx + Qdy = -\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$
(It is -C because the circle is oriented in a clockwise direction.)

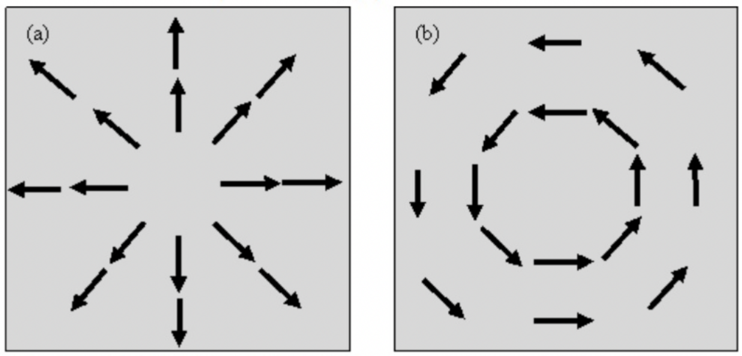
$$Q = x\sin y, P = 3y^2 - \cos y$$

$$-\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = -\iint_D \sin y - (6y + \sin y) dA = \iint_D 6y dA$$

Use polar coordinates:

$$\int_0^{\pi} \int_0^{2} (6r\sin\theta) r dr d\theta = \int_0^{\pi} 16\sin\theta d\theta = 16[\cos(0) - \cos(\pi)] = \boxed{32}$$

2. The graph below shows two vector fields. Answer the following questions for each of them.
   (1) Is it a conservative vector field?
   (2) Does it have a positive, negative, or zero curl?
   (3) Does it have a positive, negative, or zero divergence?



For vector field (a): (1) It is a conservative vector field because the line integral along any closed path is 0. (2) It has a zero curl because the vectors are not rotating. (3) It has a positive divergence because the vectors have the tendency to diverge out from a point.

For vector field (b): (1) It is not a conservative vector field because the line integral along the closed path is nonzero. (2) It has a positive curl as vectors are rotating in the counterclockwise direction. (3) It has a zero divergence because the vectors are not diverging from a single point.

3. Evaluate the flux of the vector field $F(x, y, z) = \langle x, y, xy \rangle$ where the surface S is part of the paraboloid $z = 4 - x^2 - y^2$ that lies within $0 \leq x \leq 1, 0 \leq y \leq 1$, and is oriented upwards.

The surface can be represented in the vector form:

$$\vec{r}(u, v) = \langle u, v, 4 - u^2 - v^2 \rangle,\ 0 \leq u \leq 1,\ 0 \leq v \leq 1$$
$$\vec{r_u} = \langle 1, 0, -2u \rangle$$
$$\vec{r_v} = \langle 0, 1, -2v \rangle$$
$$\vec{r_u} \times \vec{r_v} = \langle 2u, 2v, 1 \rangle \rightarrow \text{orients upwards!}$$
$$\vec{F} = \langle u, v, uv \rangle$$

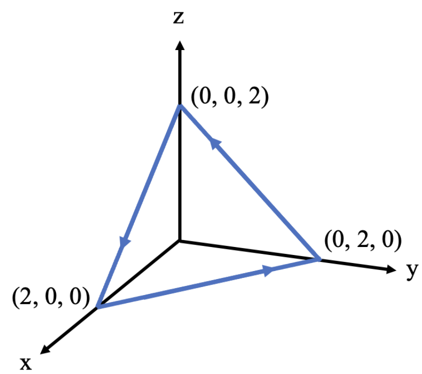
The flux can can then be evaluated as:

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r_u} \times \vec{r_v}) dA$$

$$= \int_0^1 \int_0^1 \langle u, v, uv \rangle \cdot \langle 2u, 2v, 1 \rangle dxdy = \int_0^1 \int_0^1 2u^2 + 2v^2 + uvdudv$$

$$= \int_0^1 \frac{2}{3} + 2v^2 + \frac{v}{2} dv = \boxed{\frac{19}{12}}$$

4. Use Stokes' theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle y^2, x, z \rangle$ across the curve C shown in the figure below.



Stokes' theorem states that $\int_C \vec{F} \cdot d\vec{r} = \iint_S curl\vec{F} \cdot \vec{n} dS = \iint_D curl\vec{F} \cdot (\vec{r_u} \times \vec{r_v}) dA$

$$curl\vec{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x & z \end{vmatrix} = \langle 0, 0, 1 - 2y \rangle$$

Parameterize the surface using $x = u$ and $y = v$:

$$\vec{r}(u, v) = \langle u, v, 2 - u - v \rangle$$

$$\vec{r_u} = \langle 1, 0, -1 \rangle$$

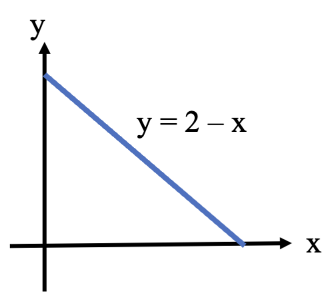
$$\vec{r_v} = \langle 0, 1, -1 \rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle 1, 1, 1 \rangle$$

Note that the direction of the cross product matches with the direction of the curve. Then, using the parameterization

$$curl\vec{F} = \langle 0, 0, 1 - 2v \rangle$$

The bounds of u and v can be determined by looking at the xy-plane shown below.



Looking at the region, $0 \leq y \leq 2 - x$ and $0 \leq x \leq 2$. Therefore, $0 \leq v \leq 2 - u$ and $0 \leq u \leq 2$. Putting everything back together gives us:

$$\iint_D curl\vec{F} \cdot (\vec{r_u} \times \vec{r_v}) dA = \int_0^2 \int_0^{2-u} \langle 0, 0, 1 - 2v \rangle \cdot \langle 1, 1, 1 \rangle dv du$$

$$= \int_0^2 \int_0^{2-u} 1 - 2v dv du = \int_0^2 2 - u - (2-u)^2 du = \int_0^2 -2 + 3u - u^2 du$$

$$= -2u + \frac{3}{2}u^2 - \frac{1}{3}u^3 \Big|_0^2 = \boxed{\frac{-2}{3}}$$