The Grainger College of Engineering
Center for Academic Resources in Engineering

MATH 241

Midterm 4 Review

Keep in mind that this presentation was created by CARE tutors, and while it is thorough, it is not comprehensive.

QR Code to the Queue

The queue contains the worksheet and the solution to this review session

Conservative Vector Field

- Line integrals of a conservative vector field are independent of path

$$\int_C F \cdot dr \text{ is independent of path D if and only if}$$

$$\int_C F \cdot dr = 0 \text{ for every closed path C in D}$$

- Let F = P**i** + Q**j** be a vector field on an open simply-connected region D. Suppose that P and Q have continuous partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

, then F is conservative.

Green's Theorem

- Let C be a **counterclockwise**, **simple closed curve** in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

- Green's theorem to calculate the area of a region D bounded by C

$$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$$

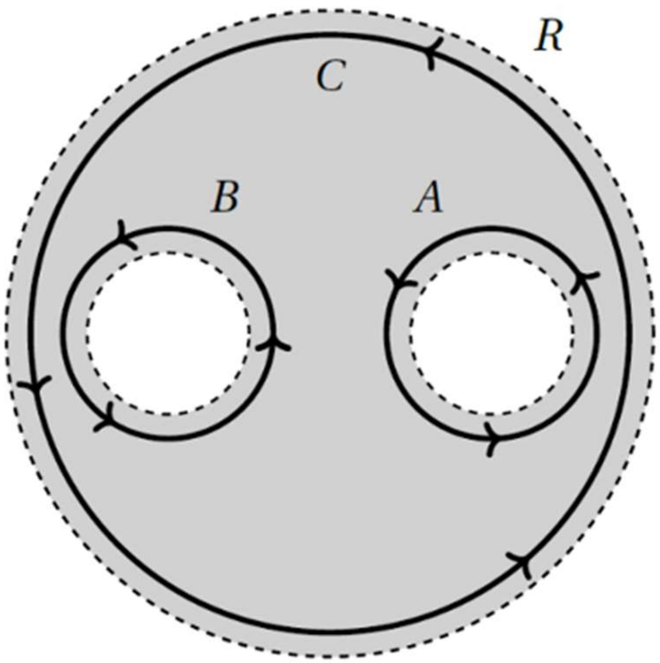
Example Question #2

- **Consider the region R shown at the right which contains simple closed curves A, B, and C. Suppose F = <P, Q> is a vector field with continuous partial derivatives on R with the following characteristics:**

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \qquad \int_A F \cdot dr = 2 \qquad \int_B F \cdot dr = -1$$

(a) Find $\int_C F \cdot dr$

(b) Is this vector field conservative?



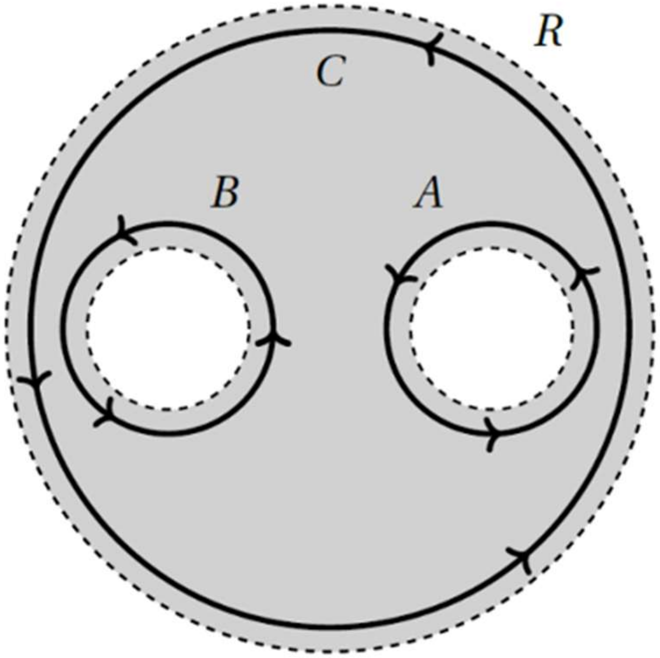
Example Solution #2

(a) Let D be the region enclosed by C.
Using Green's theorem:

$$\int_C F \cdot dr - \int_A F \cdot dr - \int_B F \cdot dr = 0$$

$$\int_C F \cdot dr - 2 - (-1) = 0$$

$$\boxed{\int_C F \cdot dr = 1}$$



(b) This vector field is not conservative because it is not a simply-connected region, and the line integral for the closed curve C is not 0.

Curl

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F}$$

- Cross product → Curl is a **vector field**
- Describes how vectors **rotate** around a certain point
- Use **right-hand rule** to determine the sign of curl
- Curl of a gradient field = 0
- If F is conservative, curl = 0
- Green's theorem in vector form:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl}\,\mathbf{F}) \cdot \mathbf{k}\, dA$$

Curl Test for Conservative Vector Field

- If F is a vector field defined on all of R^3 whose component functions have **continuous partial derivatives** and **curl F = 0**, then F is a conservative vector field

Divergence

$$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F}$$

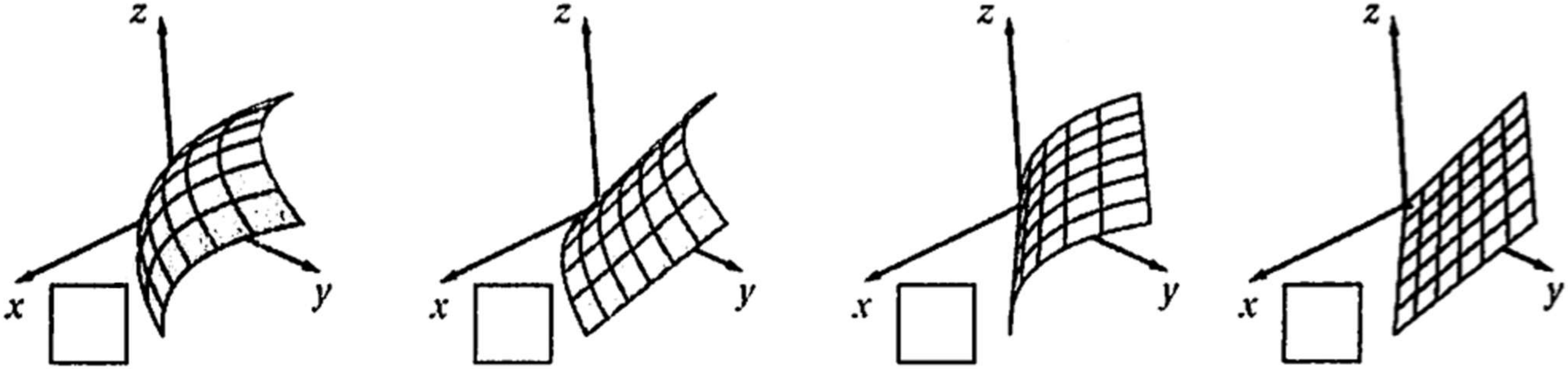
- Dot product → Divergence is a **scalar** field
- Describes how vectors diverge from a single point (or converge to a point)
- **Diverging vectors: positive, Converging vectors: negative**
- Green's theorem in vector form:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \text{div}\,\mathbf{F}(x, y)\, dA$$

Example Problem #3

- **Match the surfaces below with the following parametrization:**

$$r(u,v) = < u, u^2 + v^2, v > \text{ defined on } D = \{(u,v) | 0 \le u \le 1, 0 \le v \le 1\}$$



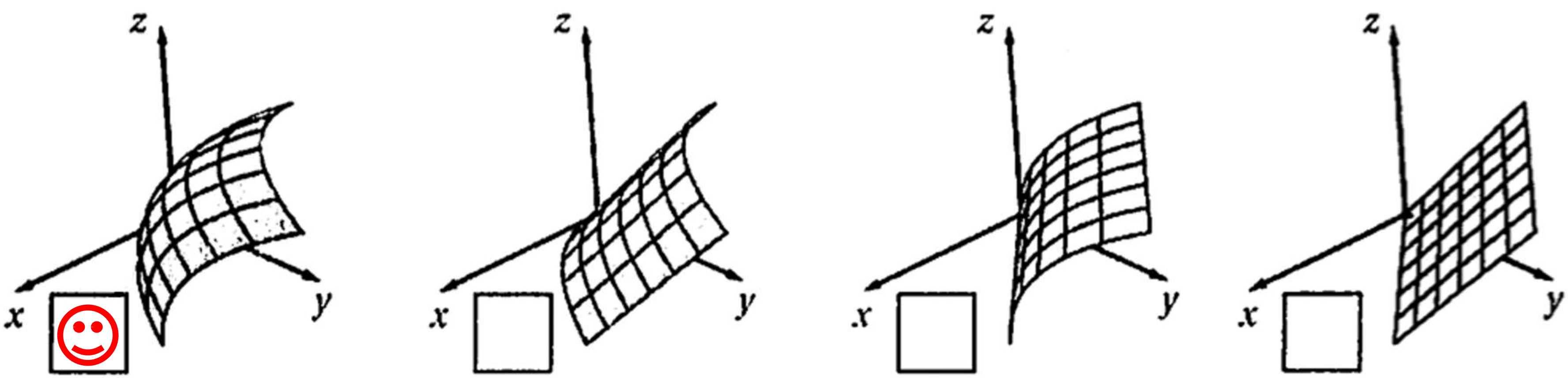
Example Solution #3

$r(u,v) = < u, u^2 + v^2, v >$ defined on $D = \{(u,v) | 0 \leq u \leq 1, 0 \leq v \leq 1\}$

When x is constant → curve on the yz-plane should be a parabola

When y is constant → curve on the xz-plane should be a circle

When z is constant → curve on the xy-plant should be a parabola



Surface Area of a Parametric Surface

- If a parametric surface S is given by the equation

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} \qquad (u, v) \in D$$

, the surface area of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

, where r_u and r_v are partial derivatives with respect to u and v.

Surface Integral

- The surface integral of a function f over a parametric surface is:

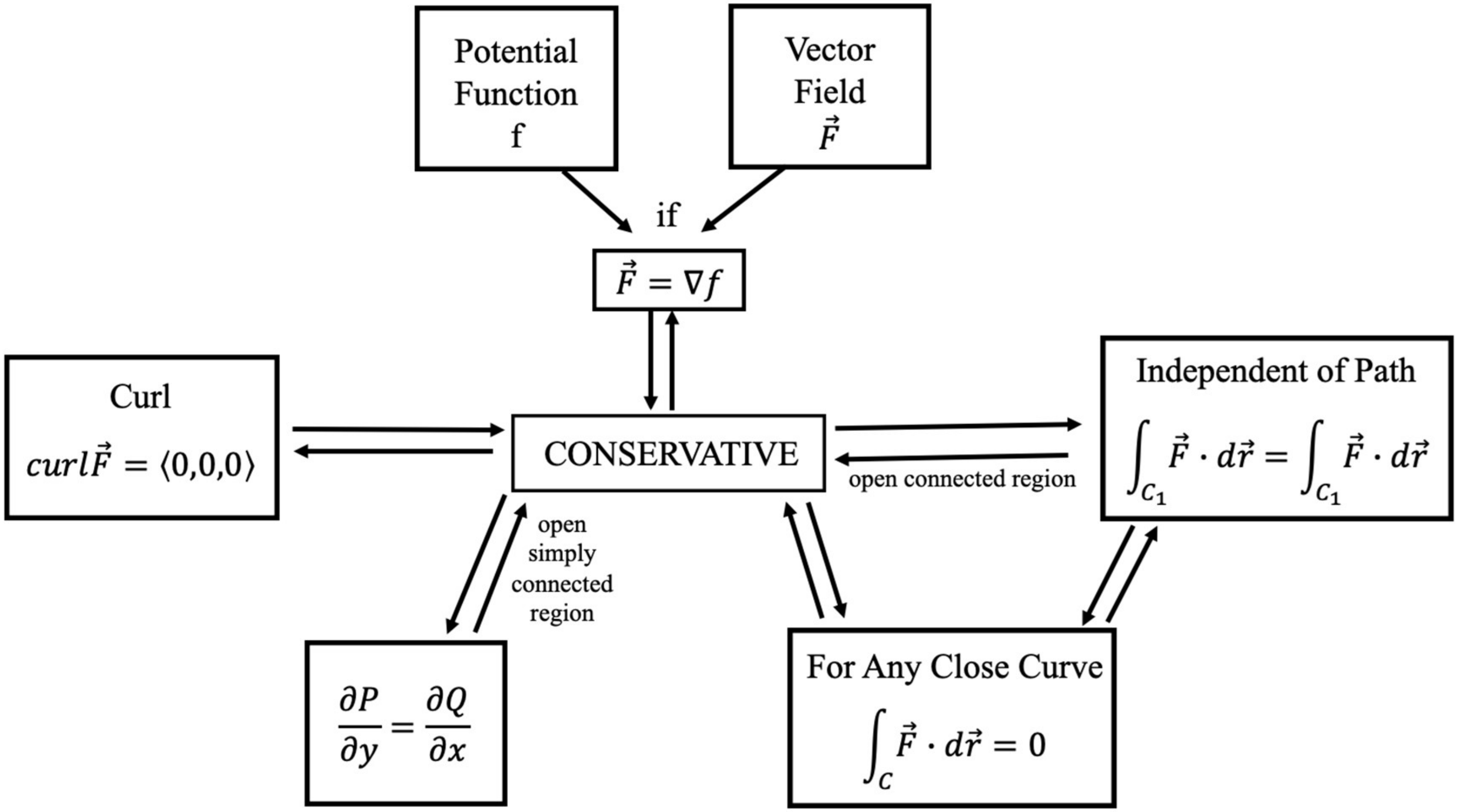
$$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

Flux

- The flux of a vector field F over a parametric surface is:

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_D \vec{F} \cdot (\overrightarrow{r_u} \times \overrightarrow{r_v})\, dA$$

Conservative Vector Field



Stokes' Theorem

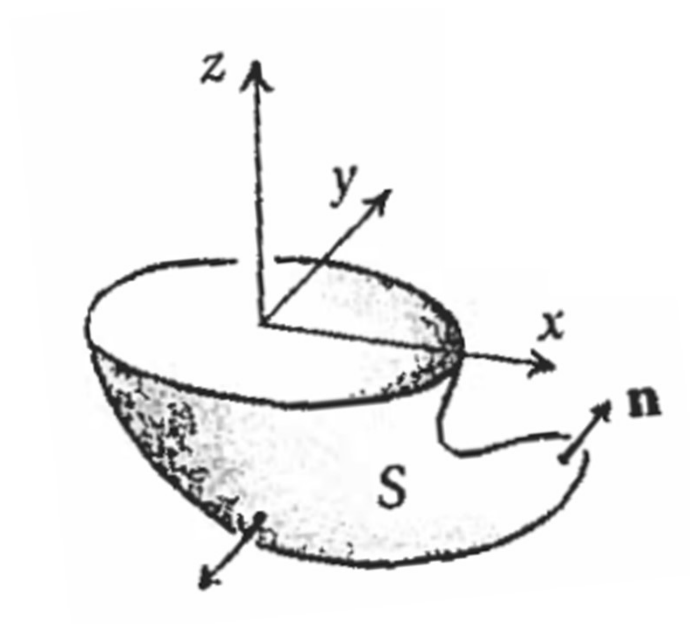
- Let S be a surface that is bounded by a **simple, counterclockwise** boundary C, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS$$

- For a conservative vector field, curl F = 0 → Line integral = 0

Example Question #1

- Calculate $\iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S}$ of F = <y+z, –x, yz> across the surface S, which has a boundary of a unit circle on the xy-plane.



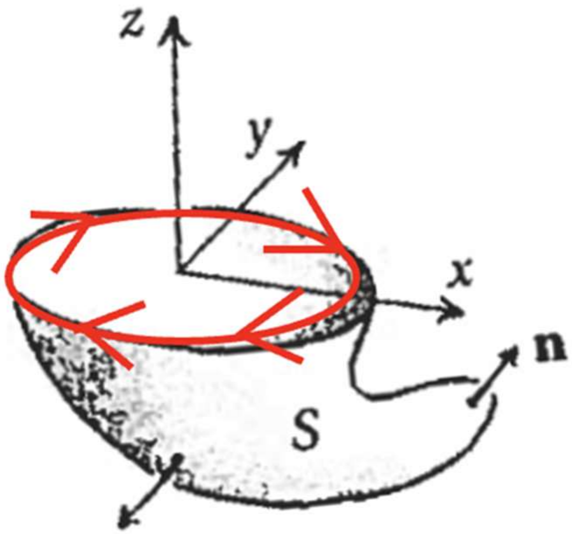
Example Solution #1

- **Use Stokes' theorem** $\iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$
- **Use the unit circle boundary as the curve C (oriented clockwise due to the direction of the normal vector → right hand rule)**

$$\vec{r} = < sint, cost, 0 >, \vec{F} = < cost, -sint, 0 >$$

$$d\vec{r} = < cost, -sint, 0 >$$

$$\int_0^{2\pi} cos^2t + sin^2t\, dt = \boxed{2\pi}$$



Divergence Theorem

- Let E be a simple solid region and let S be the **boundary surface of E with positive orientation**, then the flux of the vector field is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV$$

- If divF > 0, the vector field has a positive flux
  - Vectors are pointing outwards → **source**
- If divF < 0, the vector field has a negative flux
  - Vectors are pointing inwards → **sink**