MATH 257 Exam 3 CARE Review

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In–Person Resources

CARE Drop-in tutoring:
7 days a week on the 4th floor of Grainger Library!
Sunday - Thursday 12pm-10pm
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Course Office hours:
TAs: Mondays - Thursdays 5-7pm
Mondays: Loomis 136
Tuesdays: Transportation Building 203
Wednesdays: Loomis 136
Thursdays: Loomis 139
Instructors: Chuang (PL1): M 3-5PM in CAB 233
Luecke (PL2,PL3): Tu 3:30-5:30 in Altgeld 105

Subject	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
Math 257	1pm-5pm 6pm-9pm	2pm-10pm	12pm-2pm 3pm-10pm	12pm-2pm 4pm-6pm 7pm-10pm	1pm-7pm 8pm-10pm	1pm-5pm	12pm-2pm 3pm-5pm

Topic Summary

- Linear Transformation
- Coordinate Matrices
- Determinants
- Eigenvectors and eigenvalues
- Markov Matrices
- Diagonalization
- Matrix powers
    - Matrix exponential
- Linear differential equations

Linear Transformations

Definition. Let V and W be vector spaces. A map $T : V \to W$ is a **linear transformation** if

$$T(a\mathbf{v} + b\mathbf{w}) = aT(\mathbf{v}) + bT(\mathbf{w})$$

for all $\mathbf{v}, \mathbf{w} \in V$ and all $a, b \in \mathbb{R}$.

Theorem 50. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there is a $m \times n$ matrix A such that*

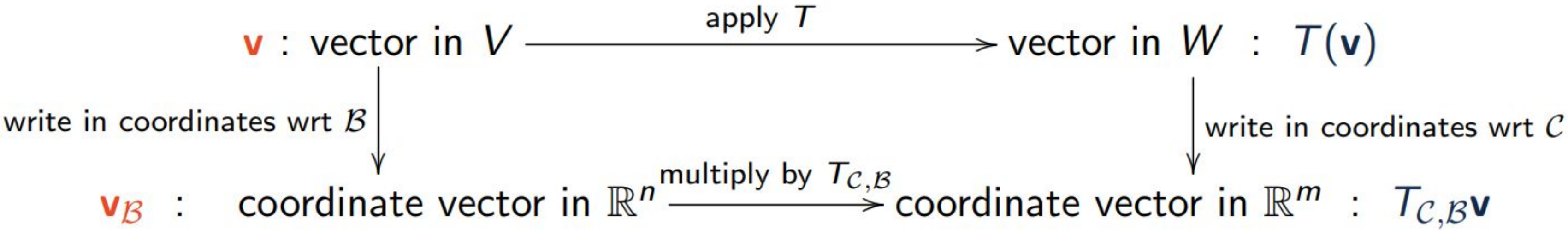
- $T(\mathbf{v}) = A\mathbf{v}, \quad$ *for all* $\mathbf{v} \in \mathbb{R}^n$.
- $A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \end{bmatrix}$, *where* $(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)$ *is the standard basis of* $\mathbb{R}^n$.

Remark. We call this A the **coordinate matrix of** T with respect to the standard bases - we write $T_{\mathcal{E}_m,\mathcal{E}_n}$.

Coordinate matrices

Theorem 51. *Let V, W be two vector space, let $\mathcal{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ be a basis of V and $\mathcal{C} = (\mathbf{c}_1, \ldots, \mathbf{c}_m)$ be a basis of W, and let $T : V \to W$ be a linear transformation. Then there is a $m \times n$ matrix $T_{\mathcal{C},\mathcal{B}}$ such that*

- $T(\mathbf{v})_{\mathcal{C}} = T_{\mathcal{C},\mathcal{B}} \mathbf{v}_{\mathcal{B}}, \quad$ *for all* $\mathbf{v} \in V$.
- $T_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} T(\mathbf{b}_1)_{\mathcal{C}} & T(\mathbf{b}_2)_{\mathcal{C}} & \ldots & T(\mathbf{b}_n)_{\mathcal{C}} \end{bmatrix}$.



Determinants (how to find them)

2x2: easy formula!

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Triangular: multiply all of the diagonal entries together

Otherwise: cofactor expansion

Note: if the matrix A is not invertible, det(A) = 0 ← this is the definition of a determinant!

Cofactor Expansion

Definition. Let A be an $n \times n$-matrix. The **(i, j)-cofactor** of A is the scalar C_{ij} defined by

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$

Procedure for large matrices:

- Pick one row or one column to eliminate
- Go one by one in the other dimension (row or column) and ignore all the entries in that row + column
    - Calculate the cofactor
    - Find the determinant of the remaining matrix

This is very impractical for anything larger than 3x3!

Cofactor Expansion Example

$$\begin{vmatrix}1&2&0\\3&-1&2\\2&0&1\end{vmatrix} = 2\cdot(-1)^{1+2}\cdot\begin{vmatrix}3&2\\2&1\end{vmatrix} + (-1)\cdot(-1)^{2+2}\cdot\begin{vmatrix}1&0\\2&1\end{vmatrix} + 0\cdot(-1)^{3+2}\cdot\begin{vmatrix}1&0\\3&2\end{vmatrix}$$

$$= -2\cdot(-1) + (-1)\cdot 1 - 0 = 1$$

$$\begin{vmatrix}1&2&0\\3&-1&2\\2&0&1\end{vmatrix} = 0\cdot(-1)^{1+3}\cdot\begin{vmatrix}3&-1\\2&0\end{vmatrix} + 2\cdot(-1)^{2+3}\cdot\begin{vmatrix}1&2\\2&0\end{vmatrix} + 1\cdot(-1)^{3+3}\cdot\begin{vmatrix}1&2\\3&-1\end{vmatrix}$$

$$= 0 - 2\cdot(-4) + 1\cdot(-7) = 1$$

Properties of determinants

(Replacement) Adding a multiple of one row to another row *does not change* the determinant.
(Interchange) Interchanging two different rows *reverses the sign* of the determinant.
(Scaling) Multiplying all entries in a row by s, *multiplies* the determinant by s.

These three things also apply to the columns of a matrix!

Let A, B be two $n \times n$-matrices. Then $\det(AB) = \det(A)\det(B)$

If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$

Let A be an $n \times n$-matrix. Then $\det(A^T) = \det(A)$

Eigenvectors and Eigenvalues

An **eigenvector** of A is a nonzero $\mathbf{v} \in \mathbb{R}^n$

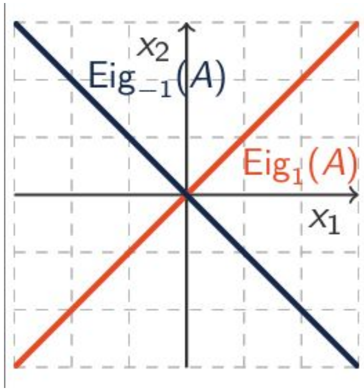
such that

$$A\mathbf{v} = \lambda \mathbf{v}$$

eigenvalue

An eigenspace is all the eigenvectors associated with a specific eigenvalue.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



$$A \begin{bmatrix} x \\ x \end{bmatrix} = 1 \cdot \begin{bmatrix} x \\ x \end{bmatrix}$$

$$A \begin{bmatrix} -x \\ x \end{bmatrix} = -1 \cdot \begin{bmatrix} -x \\ x \end{bmatrix}$$

Eigenvectors are always linearly independent!

Calculating eigenvectors and eigenvalues

Theorem 59. *Let A be an $n \times n$ matrix. Then $p_A(t) := \det(A - tI)$ is a polynomial of degree n. Thus A has at most n eigenvalues.*

Definition. We call $p_A(t)$ the **characteristic polynomial** of A.

The roots of the characteristic polynomial are the eigenvalues

Let A be $n \times n$ matrix and let λ be eigenvalue of A. Then

$$\mathrm{Eig}_\lambda(A) = \mathrm{Nul}(A - \lambda I).$$

General algorithm: 1) find det(A-λI) and solve for λ
2) plug each eigenvalue back into A-λI
3) solve for the nullspace

Eigenvalue/eigenvector example

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 2 & 3 \\ 0 & 6-\lambda & 10 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (3-\lambda)(6-\lambda)(2-\lambda)$$

$\leadsto$ A has eigenvalues 2, 3, 6. The eigenvalues of a triangular matrix are its diagonal entries.

$$\lambda_1 = 2: \qquad A - 2I = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 10 \\ 0 & 0 & 0 \end{bmatrix} \underset{RREF}{\leadsto} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2.5 \\ 0 & 0 & 0 \end{bmatrix} \leadsto \mathrm{Nul}(A - 2I) = \mathrm{span}\left(\begin{bmatrix} 2 \\ -5/2 \\ 1 \end{bmatrix} \right)$$

$$\lambda_2 = 3: \qquad A - 3I = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 3 & 10 \\ 0 & 0 & -1 \end{bmatrix} \underset{RREF}{\leadsto} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \leadsto \mathrm{Nul}(A - 3I) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

$$\lambda_3 = 6: \qquad A - 6I = \begin{bmatrix} -3 & 2 & 3 \\ 0 & 0 & 10 \\ 0 & 0 & -4 \end{bmatrix} \underset{RREF}{\leadsto} \begin{bmatrix} 1 & \frac{-2}{3} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \leadsto \mathrm{Nul}(A - 6I) = \mathrm{span}\left(\begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix} \right)$$

Properties of Eigenvalues and Eigenvectors

For a 2x2 matrix:

$$p(\lambda) = \lambda^2 - \text{Tr}(A)\lambda + \det(A)$$

Multiplicity:

- **Algebraic** multiplicity is the multiplicity of λ in the characteristic polynomial
- **Geometric** multiplicity is the dimension of the eigenspace of λ

Trace: the sum of the diagonal entries of a matrix

- Tr(A) = sum of all eigenvalues
- det(A) = product of all eigenvalues

Markov Matrices

$$\begin{bmatrix} 0 & .25 & .4 \\ 1 & .25 & .2 \\ 0 & .5 & .4 \end{bmatrix}$$

Definition: a square matrix with non-negative entries where the sum of terms in each column is 1

A **probability vector** has entries that add up to 1

The λ of a Markov Matrix:

- 1 is always an eigenvalue, and the corresponding eigenvector is called **stationary**
- All other $|\lambda| \leq 1$

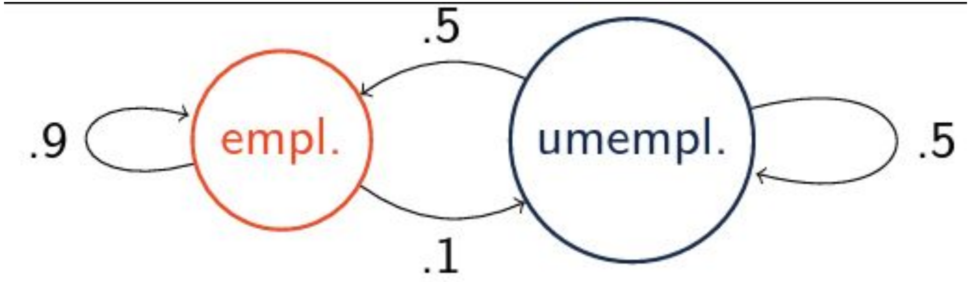
Why is a Markov Matrix useful?

Theorem 65. *Let A be an $n \times n$-Markov matrix with only positive entries and let $\mathbf{z} \in \mathbb{R}^n$ be a probability vector. Then*

$$\mathbf{z}_\infty := \lim_{k\to\infty} A^k\mathbf{z} \text{ exists,}$$

and $\mathbf{z}_\infty$ is a stationary probability vector of A (ie. $A\mathbf{z}_\infty = \mathbf{z}_\infty$).

This basically says you can left multiply A with **z** infinitely and you will get a stationary probability vector (steady state)



x_t: % of population employed at time t

y_t: % of population unemployed at time t

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} .9x_t + .5y_t \\ .1x_t + .5y_t \end{bmatrix} = \begin{bmatrix} .9 & .5 \\ .1 & .5 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

How to approach a Markov Matrix problem

1. Write out the Markov Matrix A. If it helps, make a graph like on the previous slide.
2. Determine what the question is asking you to solve for. Steady state? Intermediate state?
3. Write the probability vector of what you know of the initial state, if possible.
4. To solve for the **steady state**: Find A-1*I and solve for the nullspace, then find the probability vector in the nullspace
5. To solve for an **intermediate state**: multiply the initial state vector by the Markov matrix the appropriate number of times.

Diagonalization

$$P = \begin{bmatrix} \mathbf{v_1} & \dots & \mathbf{v_n} \end{bmatrix}$$

v are eigenvectors

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

For a matrix A to be diagonalizable:

- A must be square
- A must have as many unique eigenvectors as rows/columns (i.e. it has an eigenbasis)
- $A = PDP^{-1}$

Observe that

$$A = PDP^{-1} = I_{\mathcal{E}_n,\mathcal{B}} D I_{\mathcal{B},\mathcal{E}_n}$$

Where B is the eigenbasis → diagonalizing is a base change to the eigenbasis

Matrix Powers and Matrix Exponential

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Matrix power: diagonal matrices are easy!

$$A^m = PD^mP^{-1}$$

Where D^m =

$$\begin{bmatrix} (\lambda_1)^m & & \\ & \ddots & \\ & & (\lambda_n)^m \end{bmatrix}$$

Matrix exponential:

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \ldots$$

$$e^{At} = Pe^{Dt}P^{-1}$$

Linear Differential Equations

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}$$

With initial condition:

$$\mathbf{u}(0) = \mathbf{v}$$

Let A be an n × n matrix and $\mathbf{v} \in \mathbb{R}^n$ *The solution of the differential equation* $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$ *with initial condition* $\mathbf{u}(0) = \mathbf{v}$ *is* $\mathbf{u}(t) = e^{At}\mathbf{v}$

If $v_1, v_2, \ldots v_n$ is an eigenbasis of A:

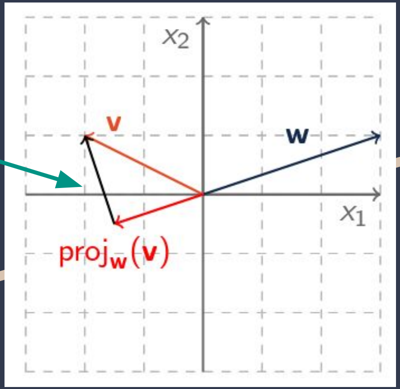
$$e^{At}\mathbf{v} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n$$

Vector Projections

Projection of **v** onto **w**

$$\text{proj}_{\mathbf{w}}(\mathbf{v}) := \frac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{w} \cdot \mathbf{w}} \mathbf{w}$$

Error term



Projecting **v** onto **w** yields the vector in span(**w**) that is closest to **v**.

The **error term** is **v** - $\text{proj}_{\mathbf{w}}(\mathbf{v})$ and is in $\text{span}(\mathbf{w})^{\perp}$

Can also use:

$$\text{proj}_{\mathbf{w}}(\mathbf{v}) = \left(\frac{1}{\mathbf{w} \cdot \mathbf{w}} \mathbf{w}\mathbf{w}^T \right) \mathbf{v}$$

Where the boxed term is called the orthogonal projection matrix onto span(**w**)

Subspace Projections

Let W *be a subspace of* $\mathbb{R}^n$ *and* $\mathbf{v} \in \mathbb{R}^n$. *Then* $\mathbf{v}$ *can be written uniquely as*

$$\mathbf{v} = \underbrace{\hat{\mathbf{v}}}_{\text{in } W} + \underbrace{\mathbf{v}^{\perp}}_{\text{in } W^{\perp}}$$

$\hat{\mathbf{v}}$ is calculated by projecting **v** onto an orthogonal basis of W

P_W is the orthogonal projection matrix for subspace W. Calculate P_W by projecting each column of the identity matrix onto W and join them all in a matrix

$Q = I - P_W$, where I is the identity. Then $P_{W^{\perp}} = Q$

Least Squares Solutions:

Trying to minimize the distance between Ax and b for an inconsistent system

$$A\hat{\mathbf{x}} = \text{proj}_{\text{Col}(A)}(\mathbf{b})$$

LSQ solution

General algorithm:

$$A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$$

Find A^T and A^TA, then solve the above system with any method you prefer.

For linear regressions:

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

A: design matrix X

$\hat{\mathbf{x}}$

b: observation vector $\mathbf{y}$

The shape of the design matrix depends on the problem!

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Questions?

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