MATH 257 Exam 3 CARE Review

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In-Person Resources

4pm-10pm 7pm-10pm

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Subject 🔶	Sunday 🔶	Monday 🔶	Tuesday 🔶	Wednesday 🔶	Thursday 🔶	Friday 🔶	Saturday 🔶
Math 257	12pm-2pm	12pm-6pm	2pm-10pm	3pm-8pm	2pm-4pm	12pm-	1pm-6pm

6pm-10pm

4pm

5pm-6pm

Topic Summary

- Linear Transformation
- Coordinate Matrices
- Determinants
- Eigenvectors and eigenvalues
- Markov Matrices

- Diagonalization
- Matrix powers
 - Matrix exponential
- Linear differential equations
- Projections
- Least Squares/ Regression

Linear Transformations

Definition. Let V and W be vector spaces. A map $T: V \to W$ is a **linear transformation** if

$$T(a\mathbf{v} + b\mathbf{w}) = aT(\mathbf{v}) + bT(\mathbf{w})$$

for all $\mathbf{v}, \mathbf{w} \in V$ and all $a, b \in \mathbb{R}$.

Theorem 50. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there is a $m \times n$ matrix A such that

$$\bullet \ T(\mathbf{v}) = A\mathbf{v}, \quad \text{for all } \mathbf{v} \in \mathbb{R}^n.$$

• $A = [T(\mathbf{e}_1) \ T(\mathbf{e}_2) \ \dots \ T(\mathbf{e}_n)]$, where $(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)$ is the standard basis of \mathbb{R}^n .

Remark. We call this A the **coordinate matrix of** T with respect to the standard bases - we write $T_{\mathcal{E}_m,\mathcal{E}_n}$.

Coordinate matrices

Theorem 51. Let V, W be two vector space, let $\mathcal{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ be a basis of V and $\mathcal{C} = (\mathbf{c}_1, \dots, \mathbf{c}_m)$ be a basis of W, and let $T : V \to W$ be a linear transformation. Then there is a $m \times n$ matrix $T_{\mathcal{C},\mathcal{B}}$ such that

$$T(\mathbf{v})_{\mathcal{C}} = T_{\mathcal{C},\mathcal{B}}\mathbf{v}_{\mathcal{B}}, \quad \text{for all } \mathbf{v} \in V.$$



Determinants (how to find them)

2x2: easy formula!

$$\det \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = ad - bc$$

Triangular: multiply all of the diagonal entries together

Otherwise: cofactor expansion

Note: if the matrix A is not invertible, $det(A) = 0 \leftarrow this$ is the definition of a determinant!

Cofactor Expansion

Definition. Let A be an $n \times n$ -matrix. The (i, j)-cofactor of A is the scalar C_{ij} defined by $C_{ij} = (-1)^{i+j} \det A_{ij}.$

Procedure for large matrices:

- Pick one row or one column to eliminate
- Go one by one in the other dimension (row or column) and ignore all the entries in that row + column
 - Calculate the cofactor
 - Find the determinant of the remaining matrix

This is very impractical for anything larger than 3x3!

Cofactor Expansion Example

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 2 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 3 & - & 2 \\ 3 & 2 & 1 \end{vmatrix} + (-1) \cdot (-1)^{2+2} \cdot \begin{vmatrix} 1 & - & 0 \\ - & + & - \end{vmatrix} + 0 \cdot (-1)^{3+2} \cdot \begin{vmatrix} 1 & - & 0 \\ 3 & - & 2 \end{vmatrix}$$

$$= -2 \cdot (-1) + (-1) \cdot 1 - 0 = 1$$

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 0 \cdot (-1)^{1+3} \cdot \begin{vmatrix} 4 & -1 \\ 3 & -1 \\ 2 & 0 \end{vmatrix} + 2 \cdot (-1)^{2+3} \cdot \begin{vmatrix} 1 & 2 \\ -1 \\ 2 & 0 \end{vmatrix} + 1 \cdot (-1)^{3+3} \cdot \begin{vmatrix} 1 & 2 \\ 3 & -1 \\ -1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix}$$

Properties of determinants

(**Replacement**) Adding a multiple of one row to another row *does not change* the determinant.

(Interchange) Interchanging two different rows *reverses the sign* of the determinant. (Scaling) Multiplying all entries in a row by *s*, *multiplies* the determinant by *s*.

These three things also apply to the columns of a matrix!

Let A, B be two $n \times n$ -matrices. Then det(AB) = det(A) det(B)If A is invertible, then $det(A^{-1}) = \frac{1}{det(A)}$

Let A be an $n \times n$ -matrix. Then $det(A^T) = det(A)$

Eigenvectors and Eigenvalues

An eigenvector of A is a nonzero $\mathbf{v} \in \mathbb{R}^n$

such that

 $A\mathbf{v} = \lambda \mathbf{v}$

eigenvalue

An eigenspace is all the eigenvectors associated with a specific eigenvalue.



Eigenvectors are always linearly independent!

Calculating eigenvectors and eigenvalues

Theorem 59. Let A be an $n \times n$ matrix. Then $p_A(t) := \det(A - tI)$ is a polynomial of degree n. Thus A has at most n eigenvalues.

Definition. We call $p_A(t)$ the characteristic polynomial of A.

The roots of the characteristic polynomial are the eigenvalues Let A be $n \times n$ matrix and let λ be eigenvalue of A. Then

$$\operatorname{Eig}_{\lambda}(A) = \operatorname{Nul}(A - \lambda I).$$

General algorithm: 1) find det(A-λI) and solve for λ
2) plug each eigenvalue back into A-λI
3) solve for the nullspace

Eigenvalue/eigenvector example

$$\det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & 2 & 3 \\ 0 & 6 - \lambda & 10 \\ 0 & 0 & 2 - \lambda \end{vmatrix} = (3 - \lambda)(6 - \lambda)(2 - \lambda)$$

 \rightarrow A has eigenvalues 2, 3, 6. The eigenvalues of a triangular matrix are its diagonal entries.

$$\lambda_{1} = 2: \qquad A - 2I = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 10 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\sim}_{RREF} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2.5 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\sim} \operatorname{Nul}(A - 2I) = \operatorname{span}\left(\begin{bmatrix} 2 \\ -5/2 \\ 1 \end{bmatrix} \right)$$
$$\lambda_{2} = 3: \qquad A - 3I = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 3 & 10 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{\sim}_{RREF} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\sim} \operatorname{Nul}(A - 3I) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$
$$\lambda_{3} = 6: \qquad A - 6I = \begin{bmatrix} -3 & 2 & 3 \\ 0 & 0 & 10 \\ 0 & 0 & -4 \end{bmatrix} \xrightarrow{\sim}_{RREF} \begin{bmatrix} 1 & \frac{-2}{3} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\sim} \operatorname{Nul}(A - 6I) = \operatorname{span}\left(\begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix} \right)$$

Properties of Eigenvalues and Eigenvectors

For a 2x2 matrix:

$$p(\lambda) = \lambda^2 - \operatorname{Tr}(A)\lambda + \det(A)$$

Multiplicity:

- Algebraic multiplicity is the multiplicity of λ in the characteristic polynomial
- Geometric multiplicity is the dimension of the eigenspace of $\boldsymbol{\lambda}$

Trace: the sum of the diagonal entries of a matrix

- Tr(A) = sum of all eigenvalues
- det(A) = product of all eigenvalues

You Try!

The matrix A has the eigenvalues as given. Compute an eigenvector corresponding to each eigenvalue.

$$\lambda_1 = 6$$
 $\lambda_2 = 10$

$$A = \begin{bmatrix} -30 & 24 \\ -60 & 46 \end{bmatrix}$$

General algorithm: 1) find det(A-λI) and solve for λ
2) plug each eigenvalue back into A-λI
3) solve for the nullspace

Solutions

The matrix A has the eigenvalues as given. Compute an eigenvector corresponding to each eigenvalue.

$$A - \lambda_1 I = A - 6I = \begin{bmatrix} -36 & 24 \\ -60 & 40 \end{bmatrix}$$
$$null(A - 6I) = span\left\{ \begin{bmatrix} 1 \\ \frac{-40}{-60} \end{bmatrix} \right\} = span\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$$
$$\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\lambda_{1} = 6 \qquad \lambda_{2} = 10 \qquad A = \begin{bmatrix} -30 & 24 \\ -60 & 46 \end{bmatrix}$$
$$A - \lambda_{2}I = A - 10I = \begin{bmatrix} -40 & 24 \\ -60 & 36 \end{bmatrix}$$
$$\operatorname{null}(A - 10I) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ \frac{-36}{-60} \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$$
$$\vec{v}_{2} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Markov Matrices

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Definition: a square matrix with non-negative entries where the sum of terms in each column is 1

A **probability vector** has entries that add up to 1

The λ of a Markov Matrix:

- 1 is always an eigenvalue, and the corresponding eigenvector is called **stationary**
- All other $|\lambda| \le 1$

Why is a Markov Matrix useful?

Theorem 65. Let A be an $n \times n$ -Markov matrix with only positive entries and let $z \in \mathbb{R}^n$ be a probability vector. Then

$$\mathbf{z}_\infty := \lim_{k o \infty} A^k \mathbf{z}$$
 exists,

and \mathbf{z}_{∞} is a stationary probability vector of A (ie. $A\mathbf{z}_{\infty} = \mathbf{z}_{\infty}$).

This basically says you can left multiply A with **z** infinitely and you will get a stationary probability vector (steady state)



 x_t : % of population employed at time t y_t : % of population unemployed at time t

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} .9x_t + .5y_t \\ .1x_t + .5y_t \end{bmatrix} = \begin{bmatrix} .9 & .5 \\ .1 & .5 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

How to approach a Markov Matrix problem

- 1. Write out the Markov Matrix A. If it helps, make a graph like on the previous slide.
- 2. Determine what the question is asking you to solve for. Steady state? Intermediate state?
- 3. Write the probability vector of what you know of the initial state, if possible.
- 4. To solve for the **steady state**: Find A-1*I and solve for the nullspace, then find the probability vector in the nullspace
- 5. To solve for an **intermediate state**: multiply the initial state vector by the Markov matrix the appropriate number of times.

Diagonalization

$$P = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$$

v are eigenvectors



For a matrix A to be diagonalizable:

- A must be square
- A must have as many unique eigenvectors as rows/columns (i.e. it has an eigenbasis)
- $A = PDP^{-1}$

Observe that

$$A = PDP^{-1} = I_{\mathcal{E}_n, \mathcal{B}} DI_{\mathcal{B}, \mathcal{E}_n}$$

Where B is the eigenbasis \rightarrow diagonalizing is a base change to the eigenbasis

Matrix Powers and Matrix Exponential

$$e^{x} = 1 + \frac{x}{1!} + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \cdots$$

Matrix power: diagonal matrices are easy! $A^m = PD^mP^{-1}$ Where $D^m = \begin{bmatrix} (\lambda_1)^m & & \\ & \ddots & \\ & & (\lambda_n)^m \end{bmatrix}$ Matrix exponential:

 $e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots$ $e^{At} = Pe^{Dt}P^{-1}$

Linear Differential Equations

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}$$

With initial condition: $\mathbf{u}(0) = \mathbf{v}$ Let A be an $n \times n$ matrix and $\mathbf{v} \in \mathbb{R}^n$ The solution of the differential equation $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$ with initial condition $\mathbf{u}(0) = \mathbf{v}$ is $\mathbf{u}(t) = e^{At}\mathbf{v}$

If v_1, v_2, \dots, v_n is an eigenbasis of A: $e^{At} \mathbf{v} = c_1 e^{\lambda_1 t} \mathbf{v}_1 + \dots + c_n e^{\lambda_n t} \mathbf{v}_n$

Vector Projections

Projection of ${\bf v}$ onto ${\bf w}$

$$\mathsf{proj}_{\mathbf{w}}(\mathbf{v}) := rac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{w} \cdot \mathbf{w}}\mathbf{w}$$



Projecting **v** onto **w** yields the vector in span(**w**) that is closest to **v**.

The **error term** is **v** - $\text{proj}_w(\mathbf{v})$ and is in $\text{span}(\mathbf{w})^{\perp}$

Can also use:

$$\mathsf{proj}_{\mathbf{w}}(\mathbf{v}) = \left(\frac{1}{\mathbf{w} \cdot \mathbf{w}} \mathbf{w} \mathbf{w}^{\mathcal{T}}\right) \mathbf{v}$$

Where the boxed term is called the orthogonal projection matrix onto span(**w**)

Subspace Projections

Let W be a subspace of \mathbb{R}^n and $\mathbf{v} \in \mathbb{R}^n$. Then \mathbf{v} can be written uniquely as



 $\hat{\boldsymbol{v}}$ is calculated by projecting \boldsymbol{v} onto an orthogonal basis of W

 $\rm P_W$ is the orthogonal projection matrix for subspace W. Calculate $\rm P_W$ by projecting each column of the identity matrix onto W and join them all in a matrix

$$Q = I - P_W$$
, where I is the identity. Then $P_{W^{\perp}} = Q$

Least Squares Solutions: Trying to minimize the distance between Ax and b for an inconsistent system

$$A\hat{\mathbf{x}} = \mathsf{proj}_{\mathsf{Col}(A)}(\mathbf{b})$$

LSQ solution

General algorithm: $A^{T}A\hat{\mathbf{x}} = A^{T}\mathbf{b}$

Find A^T and A^TA , then solve the above system with any method you prefer.

For linear regressions:



The shape of the design matrix depends on the problem!

You Try!

Given the following data points, Set up the least squares equation to solve for the coefficients to create a fit function of the form $y = \alpha x + \beta ln(x) + \gamma cos(x)$

Data Points: (1, 2.576)(2, -0.345)(3, -2.393)(4, 0.087)(5, 5.018) Reminder: General algorithm:

$$A^T A \hat{\mathbf{x}} = A^T \mathbf{b}$$

For linear regressions:



Solution

Given the following data points, Set up the least squares equation to solve for the coefficients to create a fit function of the form $y = \alpha x + \beta ln(x) + \gamma cos(x)$

 Data Points:
 If there is no noise in the data, the following is a consistent system

 (1, 2.576) $A\vec{x} = \vec{b}$

 (2, -0.345) $\begin{bmatrix} 1 & ln(1) & cos(1) \\ 2 & ln(2) & cos(2) \\ 3 & ln(3) & cos(3) \\ 4 & ln(4) & cos(4) \\ 5 & ln(5) & cos(5) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 2.576 \\ -0.345 \\ -2.393 \\ 0.087 \\ 5.018 \end{bmatrix}$

Solution

Given the following data points, Set up the least squares equation to solve for the coefficients to create a fit function of the form $y = \alpha x + \beta ln(x) + \gamma cos(x)$

The system is inconsistent, so we use the LSQ: $A^T A \hat{x} = A^T \vec{y}$

$\begin{bmatrix} 1\\2\\3\\4 \end{bmatrix}$	$ln(1) \\ ln(2) \\ ln(3) \\ ln(4)$	$\begin{array}{c} \cos(1) \\ \cos(2) \\ \cos(3) \\ \cos(4) \end{array} \right ^{T}$	$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$	$ln(1) \\ ln(2) \\ ln(3) \\ ln(4)$	$cos(1) \\ cos(2) \\ cos(3) \\ cos(4) \end{cases}$	$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} =$	$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$	$ln(1) \\ ln(2) \\ ln(3) \\ ln(4)$	$\begin{array}{c} \cos(1) \\ \cos(2) \\ \cos(3) \\ \cos(4) \end{array} \right ^{T}$	$\begin{bmatrix} 2.576 \\ -0.345 \\ -2.393 \\ 0.087 \end{bmatrix}$
$\begin{bmatrix} 4\\5 \end{bmatrix}$	ln(4) ln(5)	$\begin{bmatrix} \cos(4)\\ \cos(5) \end{bmatrix}$	$\begin{bmatrix} 4\\5 \end{bmatrix}$	ln(4) ln(5)	$\begin{bmatrix} \cos(4)\\ \cos(5) \end{bmatrix}$	$\lfloor \gamma \rfloor$	$\begin{bmatrix} 4\\5 \end{bmatrix}$	ln(4) ln(5)	$\begin{bmatrix} \cos(4)\\ \cos(5) \end{bmatrix}$	$\left[\begin{array}{c} 0.087\\ 5.018\end{array}\right]$

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Questions?



Join the queue to see the worksheet!

Important Definitions

Characteristic Polynomial: $p_A(\lambda) = det(A - \lambda I) = 0$ Eigenvectors/Eigenspaces: $\vec{v}_{\lambda} \in null(A - \lambda I) = Eig_{\lambda}(A)$ Linearity: $T(\alpha \vec{a} + \beta \vec{b}) = \alpha T(\vec{a}) + \beta T(\vec{b})$ Coordinate Inverse: $I_{A\mathcal{E}} = I_{\mathcal{E}A}^{-1}$ 2x2 Determinant: $det(A_{2\times 2}) = ad - bc$ Diagonalization: $A = PDP^{-1}$ $P = [\vec{v}_1 \cdots \vec{v}_n]$ $D = diag(\lambda_1 \cdots \lambda_n)$ Linear Differential Equation Solution: $\dot{\vec{u}} = A\vec{u}$ $\vec{u}(0) = \vec{v}$ $\vec{u} = e^{At}\vec{v} = \sum_{i=1}^n c_i e^{\lambda_i t} v_i$ Linear Least Squares: $A^T A \hat{x} = A^T \vec{b} \Rightarrow \hat{x} = (A^T A)^{-1} A \vec{b}$

1D Projections:
$$proj_{\vec{w}}(\vec{v}) = \frac{\vec{w} \cdot \vec{v}}{\vec{w} \cdot \vec{w}} \vec{w}$$
 Determinant & Trace: $det(A) = \prod_{i=1}^{i} \lambda_{i}$ $tr(A) = \sum_{i=1}^{i} \lambda_{i}$