MATH 285 Midterm 2 Review

CARE

Disclaimer

- These slides were prepared by tutors that have taken Math 285. We believe that the concepts covered in these slides could be covered in your exam.
- HOWEVER, these slides are NOT comprehensive and may not include all topics covered in your exam. These slides should not be the only material you study.
- While the slides cover general steps and procedures for how to solve certain types of problems, there will be exceptions to these steps. Use the steps as a general guide for how to start a problem but they may not work in all cases.

Topics

1. Linear Independence + Wronskian
2. Linear Constant Coefficient DE's
3. Solving Particular Solutions
   1. Undetermined Coefficients
   2. Annihilators
   3. Variation of Parameters
   4. Laplace Transformations
4. Oscillations
   1. Mechanical
   2. Electrical

Linear Independence and the Wronskian

- In order to form a "complete" solution to a differential equation, we want to create a **linear combination of solutions**
- We need to have $\boldsymbol{n}$ **solution equations**, where $\boldsymbol{n}$ **is the order** of the differential equation
- The Wronskian is a tool for determining if our solutions are linearly independent

The Wronskian

$$W(y_1, y_2, \ldots, y_n)(t) = \begin{vmatrix} y_1(t) & y_2(t) & y_3(t)\ldots & y_n(t) \\ y_1'(t) & y_2'(t) & y_3'(t)\ldots & y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & y_3^{(n-1)}(t)\ldots & y_n^{(n-1)}(t) \end{vmatrix}$$

- Calculate the determinant of the matrix built with solution functions and their derivatives
- Results
  - **If $W = 0$, the solutions are linearly dependent**
  - **If $W \neq 0$, the solutions are linearly independent**

Abel's Theorem

- If the Wronskian is non-zero, then it will solve the first order linear differential equation:

$$\boldsymbol{W' + a_{n-1}(t)W = 0}$$

Linear Constant Coefficient 2nd Order ODEs

- General Form:

$$\boldsymbol{Ay'' + By' + Cy = g(t)}$$

- Solving:
  - Set up the **characteristic equation** $\boldsymbol{Ar^2 + Br + C = 0}$
  - **Solve the roots** of the characteristic equation
  - Write the solution as $\boldsymbol{y_h = C_1 e^{r_1} + C_2 e^{r_2}}$
  - Use initial conditions to **solve the constants**

Three Specific Cases:

- Two distinct, real roots (r_1, r_2):

$$\boldsymbol{y_h = C_1 e^{r_1 t} + C_2 e^{r_2 t}}$$

- One distinct, real root (r_1):

$$\boldsymbol{y_h = C_1 t e^{r_1 t} + C_2 e^{r_1 t}}$$

- Two distinct, imaginary roots $(\mathrm{a} + \mathrm{bi}, a - bi)$:

$$\boldsymbol{y_h = e^{at}(C_1 \cos(bt) + C_2 \sin(bt))}$$

Solutions to Non-homogenous Equations

- If you have a linear non-homogenous DE:

$$\mathcal{L}y = f(t)$$

- Its general solution is given by:

$$y(t) = y_{part}(t) + y_{homog}(t)$$

$y_{homog}(t)$ – homogeneous solution
$y_{part}(t)$ - particular solution

The Method of Undetermined Coefficients

- A way to solve certain **non-homogenous linear DEs**
- Can be used when f(t) is an exponential, sin or cos, or polynomial

$f(t)$	$y(t)$
$f(t) = t^k$	$y(t) = A_0 + A_1 t + A_2 t^2 \ldots A_k t^k = P_k(t)$
$f(t) = e^{\sigma t}$	$y(t) = Ae^{\sigma t}$
$f(t) = \sin \omega t$ or $f(t) = \cos \omega t$	$y(t) = A \sin \omega t + B \cos \omega t$
$f(t) = t^k \sin \omega t$ or $f(t) = t^k \cos \omega t$	$y = P_k(t) \sin \omega t + Q_k(t) \cos \omega t$
$f(t) = e^{\sigma t} \sin \omega t$ or $f(t) = e^{\sigma t} \cos \omega t$	$y(t) = A\, e^{\sigma t} \sin \omega t + B\, e^{\sigma t} \cos \omega t$
$f(t) = t^k\, e^{\sigma t}$	$y = P_k(t)\, e^{\sigma t}$
$f(t) = t^k\, e^{\sigma t} \sin \omega t$ or $f(t) = t^k\, e^{\sigma t} \sin \omega t$	$y = P_k(t)\, e^{\sigma t} \sin \omega t + Q_k(t)\, e^{\sigma t} \cos \omega t$

Using Method of Undetermined Coefficients

1. **Initial Differential Equation**: $$y'' + 6y' + 8y = e^t$$

2. **Guess** in the same form: $$y = Ae^t$$

3. **Plug into** the initial equation: $$Ae^t + 6Ae^t + 8Ae^t = e^t$$

4. **Solve for the constants:** $$15A = 1$$

5. Write the **particular solution:** $$y_p = \frac{1}{15}e^t$$

The Method of Undetermined Coefficients Contd.

- If **guess functions appear in the homogenous solution**, **multiply by the lowest power of t** such that the guess no longer solves the homogenous equation

- Example:
  - If e^t appears in the homogenous solution and $f(t) = e^t$, guess $Ate^t + Be^t$

Annihilators

- Annihilators are **another method for solving non-homogenous differential equations**
- Look for an operator that "annihilates" the right-hand side

$f(t)$	Annihilator
1	$\frac{d}{dt}$
$P_k(t)$	$\frac{d^{k+1}}{dt^{k+1}}$
e^{at}	$\frac{d}{dt} - a$
$A \sin \omega t + B \cos \omega t$	$\frac{d^2}{dt^2} + \omega^2$
$Ae^{at} \sin \omega t + Be^{at} \cos \omega t$	$(\frac{d}{dt} - a)^2 + \omega^2$
$P_k(t) \sin \omega t + Q_k(t) \cos \omega t$	$(\frac{d^2}{dt^2} + \omega^2)^{k+1}$
$P_k(t)e^{at} \sin \omega t + Q_k(t)e^{at} \cos \omega t$	$((\frac{d}{dt} - a)^2 + \omega^2)^{k+1}$

Annihilators Contd.

- How to use annihilators to solve particular solutions:
  - Solve the homogenous equation
  - Pick the right annihilator
  - Apply the annihilator to the left-hand side
  - Find the solutions to the new homogenous equation
  - Identify the solutions that are not part of the original homogenous solution
  - Plug in your guess and solve for the coefficients

Variation of Parameters

$$y_p(t) = y_2(t)\int_0^t \frac{y_1(s)f(s)}{W(s)}ds - y_1(t)\int_0^t \frac{y_2(s)f(s)}{W(s)}ds$$

W : Wronskian
y_1 and y_2: homogenous solutions
f is the non-homogenous part

- Process:
  - **Find two solutions** to the homogenous equation (could be given)
  - **Calculate the Wronskian**
  - **Plug and chug**

Using Laplace Transformations

1. **Initial Differential Equation**: $y' = e^t \qquad y(0) = 2$

2. **Apply Laplace transform** to both sides: $$sY(s) - 2 = \frac{1}{s-1}$$

3. Algebraically **isolate the Laplacian**: $$Y(s) = \frac{1}{s(s-1)} + \frac{2}{s}$$

4. **Rewrite terms** to match Laplace tables
   * in this case, partial fraction decomposition $$Y(s) = \frac{1}{s-1} + \frac{1}{s}$$

5. **Inverse Laplace transform:** $$y(t) = e^t + 1$$

Laplace Transforms Tables

Function	Laplace Transform
$f(t)$	$F(s)$
1	$\frac{1}{s}$
t^k	$\frac{k!}{s^{k+1}}$
$t^k e^{-at}$	$\frac{k!}{(s+a)^{k+1}}$
$\sin(bt)$	$\frac{b}{b^2+s^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$e^{-at}\sin(bt)$	$\frac{b}{b^2+(s+a)^2}$
$e^{-at}\cos(bt)$	$\frac{s+a}{b^2+(s+a)^2}$

$f(t)$	$F(s)$
$f(t)+g(t)$	$F(s)+G(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$\frac{d^kf}{dt^k}$	$s^kF(s)-s^{k-1}f(0)-s^{k-2}f'(0)-\ldots f^{(k-1)}(0)$
$tf(t)$	$-F'(s)$
$t^kf(t)$	$(-1)^kF^{(k)}(s)$
$e^{at}f(t)$	$F(s-a)$
$\frac{1}{t}f(t)$	$\int_s^\infty F(\sigma)d\sigma$

Mechanical Oscillators

- Derived from fundamental physics:

$$my'' + \gamma y' + ky = f(t)$$

- Can be solved as a **standard 2nd order constant coefficient DE**
- Frequently may see **"natural frequency"** $\omega_n = \sqrt{\frac{k}{m}}$

Mechanical Oscillators Contd.

- Use the radical part of the quadratic equation to assess cases:

$$\sqrt{\gamma^2 - 4mk}$$

Criteria	**Solution**	**Physical Scenario**
$\gamma^2 = 0$	• roots = ±bi • $y_h = C_1\cos(bt) + C_2\sin(bt)$	• Undamped • Oscillates forever
$\gamma^2 < 4mk$	• roots = a±bi • $y_h = e^{at}(C_1\cos(bt) + C_2\sin(bt))$	• Underdamped • Oscillations die away slowly
$\gamma^2 = 4mk$	• roots = a • $y_h = C_1te^{r_1t} + C_2e^{r_1t}$	• Critically damped • Oscillations die away quickly
$\gamma^2 > 4mk$	• roots = a±b • $y_h = C_1e^{r_1t} + C_2e^{r_2t}$	• Overdamped • Oscillations mostly die away quickly

Electrical Oscillators

- Derived from circuit laws (for series RLC circuits specifically):

$$LI'' + RI' + \frac{1}{C}I = \frac{dV(t)}{dt}$$

- **Direct analogues** can be drawn from mechanical to electrical oscillators
  - $L = m$ (inductance)
  - $R = \Upsilon$ (resistance)
  - $\frac{1}{C} = k$ (inverse capacitance)
- **Same cases and implications** as mechanical oscillators

Thanks for Coming!