

MATH 285 Midterm 2 Review

CARE

Disclaimer

- These slides were prepared by tutors that have taken Math 285. We believe that the concepts covered in these slides could be covered in your exam.
- HOWEVER, these slides are NOT comprehensive and may not include all topics covered in your exam. These slides should not be the only material you study.
- While the slides cover general steps and procedures for how to solve certain types of problems, there will be exceptions to these steps. Use the steps as a general guide for how to start a problem but they may not work in all cases.





- I. Linear Independence + Wronskian
- II. Linear Constant Coefficient DE's
- III. Solving Particular Solutions
 - I. Undetermined Coefficients
 - II. Annihilators
 - III. Variation of Parameters
 - IV. Laplace Transformations
- IV. Oscillations
 - I. Mechanical
 - II. Electrical

Linear Independence and the Wronskian

- In order to form a "complete" solution to a differential equation, we want to create a linear combination of solutions
- We need to have *n* solution equations, where *n* is the order of the differential equation
- The Wronskian is a tool for determining if our solutions are linearly independent

The Wronskian

$$W(y_1, y_2, \dots, y_n)(t) = \begin{vmatrix} y_1(t) & y_2(t) & y_3(t) \dots & y_n(t) \\ y'_1(t) & y'_2(t) & y'_3(t) \dots & y'_n(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & y_3^{(n-1)}(t) \dots & y_n^{(n-1)}(t) \end{vmatrix}$$

- Calculate the determinant of the matrix built with solution functions and their derivatives
- Results
 - If W = 0, the solutions are linearly dependent
 - If $W \neq 0$, the solutions are linearly independent

Abel's Theorem

• If the Wronskian is non-zero, then it will solve the first order linear differential equation:

$$W' + a_{n-1}(t)W = 0$$



Linear Constant Coefficient 2nd Order ODEs

• General Form:

$$Ay'' + By' + Cy = g(t)$$

- Solving:
 - Set up the characteristic equation $Ar^2 + Br + C = 0$
 - Solve the roots of the characteristic equation
 - Write the solution as $y_h = C_1 e^{r_1} + C_2 e^{r_2}$
 - Use initial conditions to **solve the constants**

Three Specific Cases:

• Two distinct, real roots (r_1, r_2) :

$$y_h = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

• One distinct, real root (r_1) :

$$y_h = C_1 t e^{r_1 t} + C_2 e^{r_1 t}$$

• Two distinct, imaginary roots (a + bi, a - bi):

$$y_h = e^{at}(C_1 \cos(bt) + C_2 \sin(bt))$$

Solutions to Non-homogenous Equations

• If you have a linear non-homogenous DE:

$$\mathcal{L}y = f(t)$$

• Its solution is given by:

$$y(t) = y_{part}(t) + y_{homog}(t)$$

 $y_{homog}(t)$ - general solution $y_{part}(t)$ - particular solution



The Method of Undetermined Coefficients

- A way to solve certain non-homogenous linear DEs
- Can be used when f(t) is an exponential, sin or cos, or polynomial

f(t)	y(t)
$f(t) = t^k$	$y(t) = A_0 + A_1 t + A_2 t^2 \dots A_k t^k = P_k(t)$
$f(t) = e^{\sigma t}$	$y(t) = A e^{\sigma t}$
$f(t) = \sin \omega t \text{ or } f(t) = \cos \omega t$	$y(t) = A\sin\omega t + B\cos\omega t$
$f(t) = t^k \sin \omega t \text{ or } f(t) = t^k \cos \omega t$	$y = P_k(t)\sin\omega t + Q_k(t)\cos\omega t$
$f(t) = e^{\sigma t} \sin \omega t \text{ or } f(t) = e^{\sigma t} \cos \omega t$	$y(t) = A e^{\sigma t} \sin \omega t + B e^{\sigma t} \cos \omega t$
$f(t) = t^k e^{\sigma t}$	$y = P_k(t) e^{\sigma t}$
$f(t) = t^k e^{\sigma t} \sin \omega t \text{ or } f(t) = t^k e^{\sigma t} \sin \omega t$	$y = P_k(t) e^{\sigma t} \sin \omega t + Q_k(t) e^{\sigma t} \cos \omega t$

Using Method of Undetermined Coefficients

1. Initial Differential Equation:	$y^{\prime\prime}+6y^{\prime}+8y=e^{t}$
2. Guess in the same form:	$y = \mathbf{A} e^t$
3. Plug into the initial equation:	$Ae^t + 6Ae^t + 8Ae^t = e^t$
4. Solve for the constants:	15A = 1
5. Write the particular solution:	$y_p = \frac{1}{15}e^t$

The Method of Undetermined Coefficients Contd.

- If guess functions appear in the homogenous solution, multiply by the lowest power of t such that the guess no longer solves the homogenous equation
- Example:
 - If e^t appears in the homogenous solution and $f(t) = e^t$, guess $Ate^t + Be^t$



Annihilators

- Annihilators are another method for solving nonhomogenous differential equations
- Look for an operator that "annihilates" the right-hand side



Annihilators Contd.

- How to use annihilators to solve particular solutions:
 - Solve the homogenous equation
 - Pick the right annihilator
 - Apply the annihilator to the left-hand side
 - Find the solutions to the new homogenous equation
 - Identify the solutions that are not part of the original homogenous solution
 - Plug in your guess and solve for the coefficients



Variation of Parameters

$$y_p(t) = y_2(t) \int_0^t \frac{y_1(s)f(s)}{W(s)} ds - y_1(t) \int_0^t \frac{y_2(s)f(s)}{W(s)} ds$$

W : Wronskian y_1 and y_2 : homogenous solutions f is the non-homogenous part

- Process:
 - Find two solutions to the homogenous equation (could be given)
 - Calculate the Wronskian
 - Plug and chug

Laplace Transformations

- Process:
 - Apply the Laplace transform to both sides of the differential equation
 - Algebraically isolate the Laplacian of your function
 - **Rewrite terms** to match LaPlace tables
 - Inverse Laplace transform both sides
 - May require partial fraction decomposition or other tricks



Laplace Transforms Tables

Function	Laplace Transform	f(t)	F(s)
f(t)	F(s)	f(t) + g(t)	F(s) + G(s)
1	$\frac{1}{s}$	f'(t)	sF(s) - f(0)
t^k	$\frac{k!}{s^{k+1}}$	f''(t)	$s^2F(s) - sf(0) - f'(0)$
$t^k e^{-at}$	$\frac{\frac{k!}{(s+a)^{k+1}}}$	$\frac{d^k f}{dt^k}$	$s^{k}F(s) - s^{k-1}f(0) - s^{k-2}f'(0) - \dots f^{(k-1)}(0)$
sin(bt)	$\frac{b}{b^2+s^2}$	tf(t)	-F'(s)
$\cos(bt)$	$\frac{s}{s^2+b^2}$	$t^k f(t)$	$(-1)^k F^{(k)}(s)$
$e^{-at}\sin(bt)$	$\frac{b}{b^2 + (s+a)^2}$	$e^{at}f(t)$	F(s-a)
$e^{-at}\cos(bt)$	$\frac{s+a}{b^2+(s+a)^2}$	$\frac{1}{t}f(t)$	$\int_{s}^{\infty} F(\sigma) d\sigma$



Mechanical Oscillators

• Derived from fundamental physics:

$$my'' + \gamma y' + ky = f(t)$$

- Can be solved as a standard 2nd order constant coefficient DE
- Frequently may see "natural frequency" $\omega_n = \sqrt{\frac{k}{m}}$

Mechanical Oscillators Contd.

• Use the radical part of the quadratic equation to assess cases:

$$\sqrt{\gamma^2-4mk}$$

Criteria	Solution	Physical Scenario
$\gamma^2 = 0$	• $r = \pm bi$ • $y_h = C_1 \cos(bt) + C_2 \sin(bt)$	UndampedOscillates forever
$\gamma^2 < 4mk$	• $r = a \pm bi$ • $y_h = e^{at}(C_1 \cos(bt) + C_2 \sin(bt))$	UnderdampedOscillations die away slowly
$\gamma^2 = 4mk$	• $r = a$ • $y_h = C_1 t e^{r_1 t} + C_2 e^{r_1 t}$	Critically dampedOscillations die away quickly
$\gamma^2 > 4mk$	• $r = a \pm b$ • $y_h = C_1 e^{r_1 t} + C_2 e^{r_2 t}$	OverdampedOscillations mostly die away quickly

Electrical Oscillators

• Derived from circuit laws (for series RLC circuits specifically):

$$LI'' + RI' + \frac{1}{C}I = \frac{dV(t)}{dt}$$

- Direct analogues can be drawn from mechanical to electrical oscillators
 - L = m (inductance)
 - $R = \Upsilon$ (resistance)
 - $\frac{1}{c} = k$ (inverse capacitance)
- Same cases and implications as mechanical oscillators

Thanks for Coming!

