The Grainger College of Engineering
Center for Academic Resources in Engineering

MATH 241

Midterm 2 Review

Keep in mind that this presentation was created by CARE tutors, and while it is thorough, it is not comprehensive.

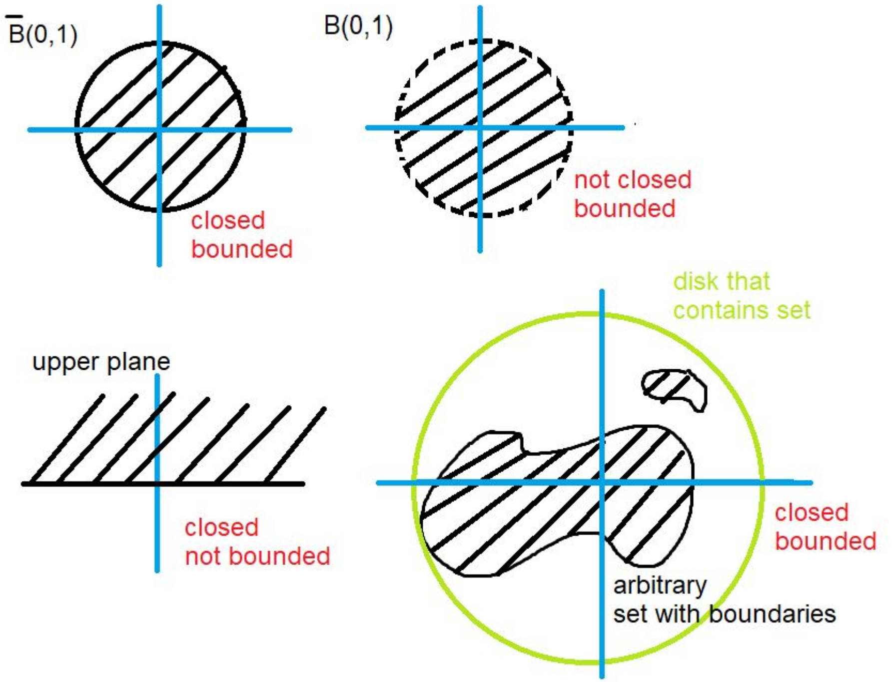
QR Code to the Queue

The queue contains the worksheet and the solution to this review session

Extreme Value Theorem

If f(x,y) is continuous on a closed and bounded set D, then it is guaranteed that f has an absolute minimum and maximum value

- The absolute min and max will either occur at the critical points of f, or on the endpoints of the boundary D



https://math.stackexchange.com/questions/1190640/what-is-the-difference-between-closed-and-bounded-in-terms-of-domains

Example Question #1

- Consider the function $f = x^3 + y^3 + 3xy$. If the critical points of f are (0, 0) and (–1, –1), classify them into local mins, maxes, and saddle points.

Example Solution #1

- Consider the function f = $x^3 + y^3 + 3xy$. If the critical points of f are (0, 0) and (–1, –1), classify them into local mins, maxes, and saddle points.

$$f_x = 3x^2 + 3y \quad f_y = 3y^2 + 3x \quad f_{xx} = 6x \quad f_{yy} = 6y$$

$$f_{xy} = f_{yx} = 3$$

$$\text{At } (0,0), D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} = -9 \rightarrow \text{Saddle Point}$$

$$\text{At } (-1,-1), D = \begin{vmatrix} -6 & 3 \\ 3 & -6 \end{vmatrix} = 27 \rightarrow \text{Because } f_{xx} = -6 < 0 \rightarrow \text{Local Max}$$

Gradient and Directional Derivatives

$$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

- The gradient will always point **perpendicular to the level curves/surfaces of f**
- $\nabla f = 0$ at a local minimum/maximum

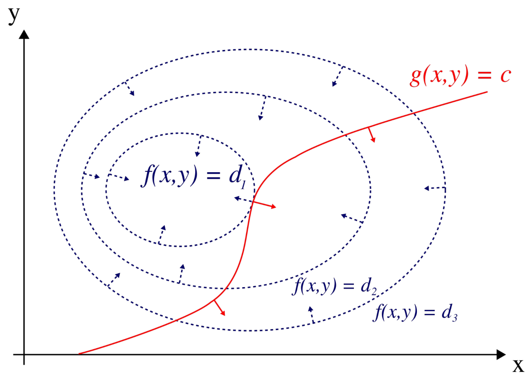
$$D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

- Tells you how the function f changes along the vector u

Lagrange Multiplier

- Solve the following system of equations for λ (Lagrange Multiplier)
  - Where f is the function, and g is the constraint

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$
$$g(x, y, z) = k$$



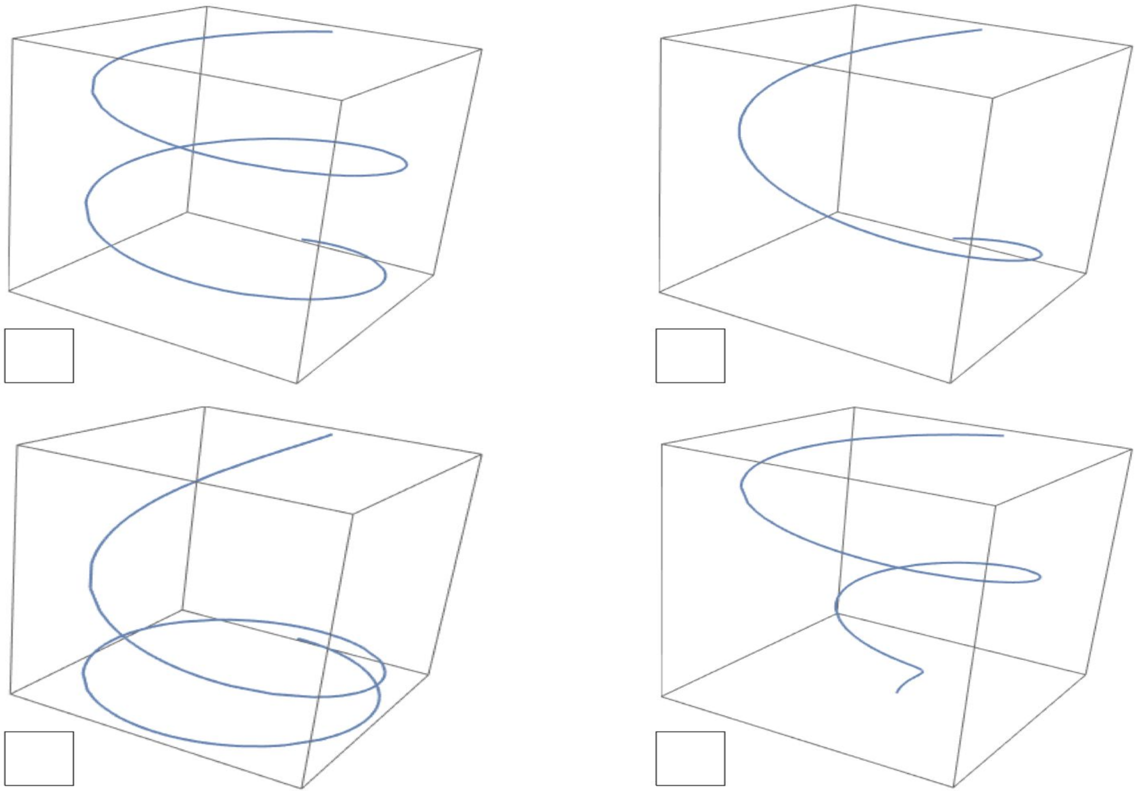
Example Question #2

Let C be the curve parameterized by **r**(t) = $\langle \sin(t^2), \cos(t^2), t^2 \rangle$ for $0 \le t \le 2\sqrt{\pi}$. Check the corresponding picture of C.

(Pictures are on the next slide)

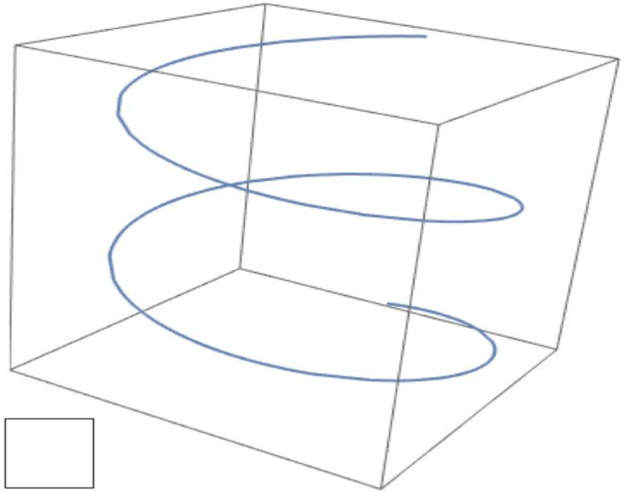
Example Q

Let C be the curve parameterized by $\mathbf{r}(t) = \langle \sin(t^2), \cos(t^2), t^2 \rangle$ for $0 \leq t \leq 2\sqrt{\pi}$. Check the corresponding picture of C.



Example Solution #2

$\mathbf{r}(t) = \langle \sin(t^2), \cos(t^2), t^2 \rangle$ for $0 \le t \le 2\sqrt{\pi}$



Example Question #3

Find the vector function representing the curve of intersection between the circular cylinder of radius 4 centered on the z-axis and the surface $z = xy$.

Example Solution #3

Find the vector function representing the curve of intersection between the circular cylinder of radius 4 centered on the z-axis and the surface z = xy.

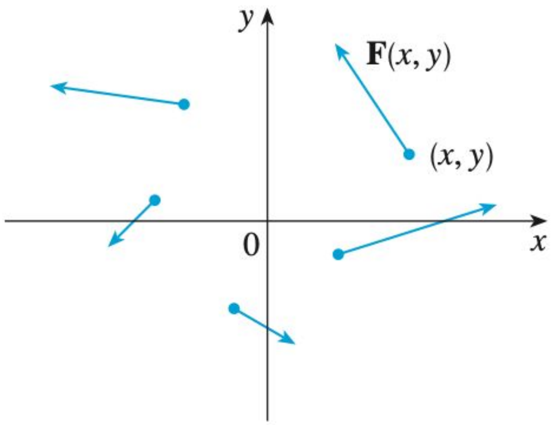
$$\overrightarrow{r_{cyl}} = \langle 4\cos t, 4\sin t \rangle$$

$$z = xy = 16\cos t \cdot \sin t$$

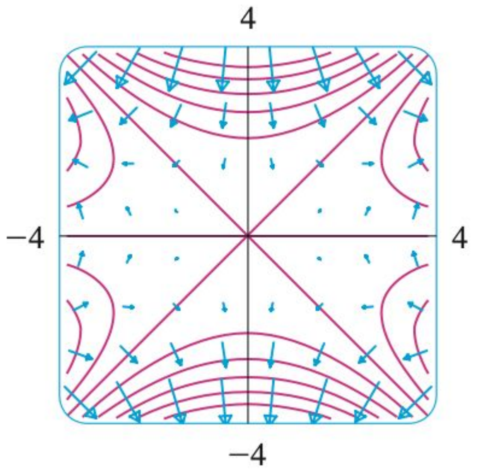
$$\vec{r}(t) = \langle 4\cos t, 4\sin t, 16\cos t \cdot \sin t \rangle$$

Vector Field, Gradient Vector Field

- A vector field F(x,y) = P **i** + Q **j** is a function that assigns each point (x,y) a 2D vector



- A gradient vector field ∇ F(x,y) is a vector field that is always **perpendicular to the contour map**



Line Integral Along a Curve with respect to...

- Arc length (orientation does not matter, **integral of C = integral of –C**)

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

- x, y (orientation matters, **integral of C = –integral of –C**)

$$\int_C f(x, y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$$

$$\int_C f(x, y)\, dy = \int_a^b f(x(t), y(t))\, y'(t)\, dt$$

Line Integral of Vector Fields

- Let F be a continuous vector field defined on a curve C given by a vector function r(t), a ≤ t ≤ b. Line integral of F along C (**Work done**) is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\, dx + Q\, dy + R\, dz$$

where $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$

Fundamental Theorem of Line Integrals

- Let C be a smooth curve given by the vector function r(t), a ≤ t ≤ b. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then:

$$\int_C \nabla f \cdot dr = f[r(b)] - f[r(a)]$$

Conservative Vector Field

- Line integrals of a conservative vector field are independent of path

$\int_C F \cdot dr$ is independent of path D if and only if

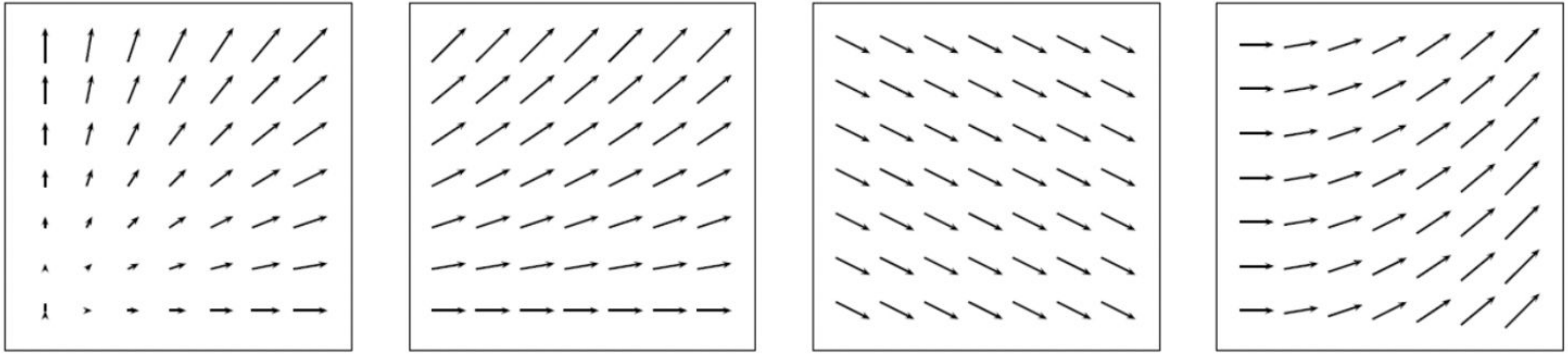
$\int_C F \cdot dr = 0$ for every closed path C in D

- Let F = P**i** + Q**j** be a vector field on an open simply-connected region D. Suppose that P and Q have continuous partial derivatives and

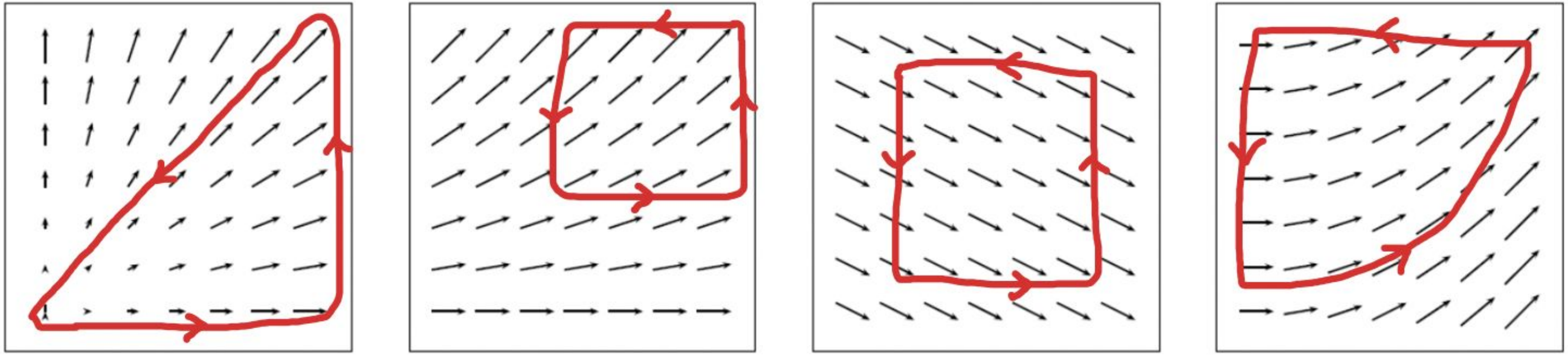
$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout D , then F is conservative.

Example Question #5

- **Which one of the vector fields shown below is not conservative?**



Example Solution #5



The fourth vector field is not conservative as line integral in the closed path does not equal to 0.