MATH 257 Exam 2 CARE Review

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In-Person Resources

CARE Drop-in tutoring:
7 days a week on the 4th floor of Grainger Library!
Sunday - Thursday 12pm-10pm
Friday & Saturday 12-6pm

Course Office hours:
TAs: Mondays - Thursdays 5-7pm
Mondays: Loomis 136
Tuesdays: Transportation Building 203
Wednesdays: Loomis 136
Thursdays: Loomis 139
Instructors: Chuang (PL1): M 3-5PM in CAB 233
Luecke (PL2,PL3): Tu 3:30-5:30 in Altgeld 105

Subject	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
Math 257	1pm-5pm 6pm-9pm	2pm-10pm	12pm-2pm 3pm-10pm	12pm-2pm 4pm-6pm 7pm-10pm	1pm-7pm 8pm-10pm	1pm-5pm	12pm-2pm 3pm-5pm

Topic Summary

- LU Decomposition
  - Lower/Upper Triangular Matrix
  - LU for Linear Systems
  - Permutation Matrix
- Orthogonal Matrices
  - Inner Product
  - Orthogonality
- Subspaces
  - Column Space
  - Null Space
  - Linear Independence
  - Fundamental Subspaces
- Basis and Dimension
  - Fundamental Subspaces
  - Orthonormal bases
- Graph and Adjacency Matrices
- Coordinates
  - Coordinate Matrices
  - Orthogonal/normal Complements

Upper/Lower Triangular Matrices

Upper Triangular:

$$\begin{bmatrix} \star & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & * \end{bmatrix}$$

Finding this is like doing REF with only row replacement

Lower Triangular:

$$\begin{bmatrix} \star & 0 & 0 & 0 & 0 \\ \star & \star & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 \\ \star & \star & \star & \ddots & \vdots \\ \star & \star & \star & \star & \star \end{bmatrix}$$

Keep track of your row operations to find L

LU Decomposition:

$A = LU$

- Not all matrices have LU decompositions
- LU decompositions are not unique (unlike inverses)

Finding the LU Decomposition

Determine the LU-decomposition of $\begin{bmatrix} 1 & 2 & 2 \\ 4 & 4 & 4 \\ 4 & 4 & 8 \end{bmatrix}$

1) Col 1 Row 2 2) Col 1 Row 3

3) Col 3 Row 3

$$\begin{bmatrix} 1 & 2 & 2 \\ 4 & 4 & 4 \\ 4 & 4 & 8 \end{bmatrix} \underset{R_2 \to R_2 \boxed{-4} R_1, R_3 \to R_3 \boxed{-4} R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 2 \\ 0 & -4 & -4 \\ 0 & -4 & 0 \end{bmatrix} \underset{R_3 \to R_3 \boxed{-1} R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 2 \\ 0 & -4 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

$$, L := \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix} \qquad U := \begin{bmatrix} 1 & 2 & 2 \\ 0 & -4 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

LU for Linear Systems

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}}_{L} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}}_{\mathbf{b}}$$

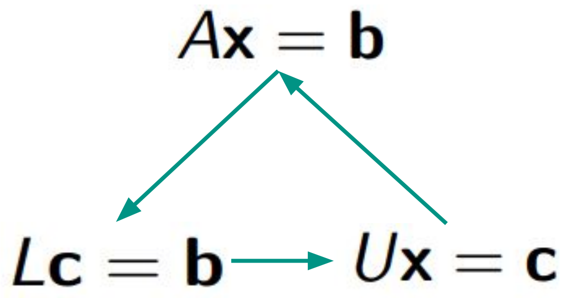
$$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix}}_{\mathbf{c}}.$$

Use **LU decomposition** to solve a linear system if:

1. A is nxn matrix
2. A = LU
3. b ∈ R^n

Step-by-step Algorithm

1. Find L and U
2. Solve for c using Lc = b
3. Solve for x using Ux = c



$$A\mathbf{x} = (LU)\mathbf{x} = L(U\mathbf{x}) = L\mathbf{c} = \mathbf{b}$$

Permutation Matrices: for matrices that don't have an LU decomposition

Theorem 21. *Let A be $n \times n$ matrix. Then there is a permutation matrix P such that PA has an LU-decomposition.*

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix} \underset{R_1 \leftrightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = PA$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Step-by-step:

- Use the interchange operation done on **A** to an equivalent size identity matrix, this will be your **P** matrix
- Solve the for the **LU** decomposition of **PA**

When we apply the $\mathbf{P}^{-1}$ to **LU** (on the right), we'll be able to get the original value of **A**

$$PA = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ .5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=:L} \underbrace{\begin{bmatrix} 2 & 1 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=:U}$$

Inner Product, Norm, and Distance

If $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$, then $\mathbf{v} \cdot \mathbf{w}$ is

$$v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

The **inner product** of $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ is

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w}$$

AKA the dot product

It is a scalar!

Definition. Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

The **norm** (or **length**) of $\mathbf{v}$ is

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + \cdots + v_n^2}.$$

The **distance** between $\mathbf{v}$ and $\mathbf{w}$ is

$$\text{dist}(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|.$$

The norm is also a scalar!

Properties of the Inner Product: similar to scalars

Theorem 22. *Let* $\mathbf{u}, \mathbf{v}$ *and* $\mathbf{w}$ *be vectors in* $\mathbb{R}^n$*, and let* c *be any scalar. Then*

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ Commutative!

(b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$ Distributive!

(c) $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$ Associative!

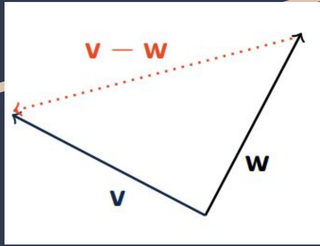
(d) $\mathbf{u} \cdot \mathbf{u} \geq \mathbf{0}$, *and* $\mathbf{u} \cdot \mathbf{u} = \mathbf{0}$ *if and only if* $\mathbf{u} = \mathbf{0}$.

Orthogonality

(fancy word for perpendicular)

Vectors are orthogonal if their **dot product is zero**.

Why? The dot product of two non-zero vectors can only be zero if the angle between them is 90.



Orthonormality

A **unit vector** in $\mathbb{R}^n$ is vector of length 1.

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

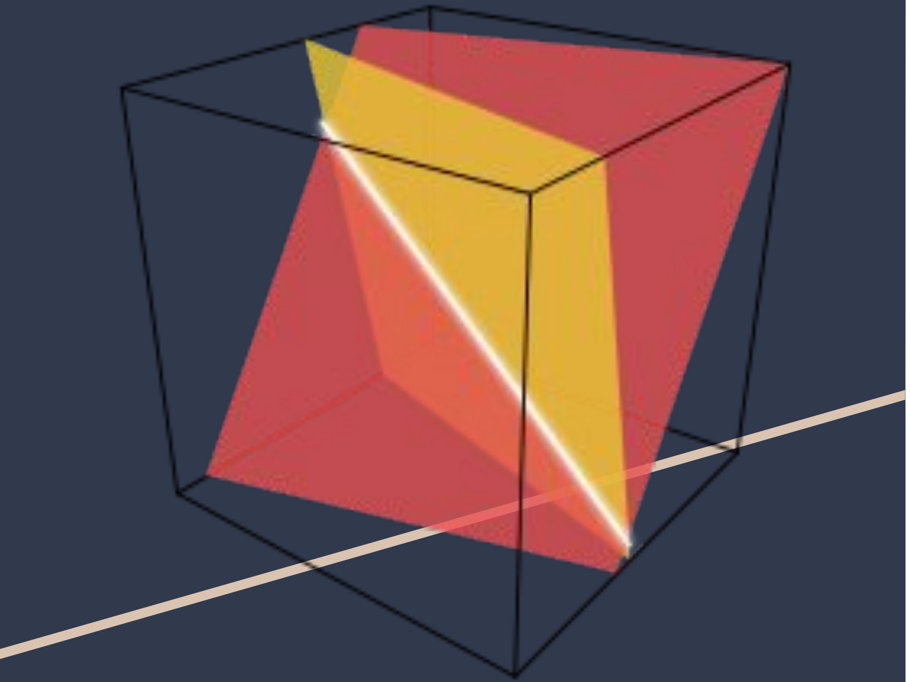
Orthonormal sets are all orthogonal to each other and unit vectors.

Ex.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Subspaces



W is a **subspace** of **V,** if:

- W contains the 0 vector
- Adding any 2 vectors in W together gives a vector also in W
- Multiplying any vector in W by any scalar gives a vector also in W

Theorem 24. *Let* $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m} \in \mathbb{R}^n$. *Then* $Span(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m})$ *is a subspace of* $\mathbb{R}^n$.

Vector Spaces ‘V’: a specific type of subspace

$u, v, w \in V$ and for all scalars $c, d \in \mathbb{R}$:

- $\mathbf{u} + \mathbf{v}$ is in V. (V is “closed under addition”.)
- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
- There is a vector (called the zero vector) $\mathbf{0}_V$ in V such that $\mathbf{u} + \mathbf{0}_V = \mathbf{u}$.
- For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V satisfying $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}_V$.
- $c\mathbf{u}$ is in V. (V is “closed under scalar multiplication”.)
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
- $(cd)\mathbf{u} = c(d\mathbf{u})$.
- $1\mathbf{u} = \mathbf{u}$.

Column Spaces

Definition. The **column space**, written as Col(A), of an $m \times n$ matrix A is the set of all linear combinations of the columns of A. If $A = \begin{bmatrix} \mathbf{a_1} & \mathbf{a_2} & \cdots & \mathbf{a_n} \end{bmatrix}$, then $\text{Col}(A) = \text{span}(\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n})$.

$$A = \begin{bmatrix} 1 & -10 & -24 & -42 \\ 1 & -8 & -18 & -32 \\ -2 & 20 & 51 & 87 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -10 & -24 & -42 \\ 1 & -8 & -18 & -32 \\ -2 & 20 & 51 & 87 \end{bmatrix} \xrightarrow[R_3 + 2R_1 \to R_3]{R_2 - R_1 \to R_2} \begin{bmatrix} \underline{1} & -10 & -24 & -42 \\ 0 & \underline{2} & 6 & 10 \\ 0 & 0 & \underline{3} & 3 \end{bmatrix}$$

$$\text{Col}(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -10 \\ -8 \\ 20 \end{bmatrix}, \begin{bmatrix} -24 \\ -18 \\ 51 \end{bmatrix} \right\}$$

How to solve for Col(A):

1. Put matrix A into REF
2. Find all the pivots of A
3. Map the pivots to the columns of your original matrix, A

Null Spaces

Definition. The **nullspace** of an $m \times n$ matrix A, written as Nul(A), is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$; that is, $\text{Nul}(A) = \{\mathbf{v} \in \mathbb{R}^n \ : \ A\mathbf{v} = \mathbf{0}\}$.

How to solve for Nul(A):

1. Set matrix A into **Augmented Matrix** with zeros on the right (**Ax = 0**)
2. Get A into **RREF**
3. Solve for **x**

Nul(A) = span($\mathbf{x}_1$, $\mathbf{x}_2$,...)

Null Space Example

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

RREF →

$$A = \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- $x_1 - 2x_2 - x_4 + 3x_5 = 0$
- $x_3 + 2x_4 - 2x_5 = 0$

$$x_1 = 2x_2 + x_4 - 3x_5$$
$$x_3 = -2x_4 + 2x_5$$

$$\begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\text{Nul}\,A = \text{Span}\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Linear Independence

Definition. Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are said to be **linearly independent** if the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution (namely, $x_1 = x_2 = \cdots = x_p = 0$).
We say the vectors are **linearly dependent** if they are not linearly independent.

Theorem 30. *Let A be an $m \times n$ matrix. The following are equivalent:*

- *The columns of A are linearly independent.*
- *$A\mathbf{x} = \mathbf{0}$ has only the solution $\mathbf{x} = \mathbf{0}$.*
- *A has n pivots.*
- *there are no free variables for $A\mathbf{x} = \mathbf{0}$.*

Basis and Dimension

Definition. Let V be a vector space. A sequence of vectors $(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ in V is a **basis** of V if

- $V = \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_p)$, and
- $(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ are linearly independent.

The number of vectors in a basis of V is the **dimension** of V.

Basis and Dimension example

Is $\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right)$ a basis of $\mathbb{R}^3$?

Our set has 3 vectors and

$$\dim \mathbb{R}^3 = 3$$

General Definition of dimension

$$\dim \mathbb{R}^n = n$$

Next, we check linear independence

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

*Recall: All pivots for **nxn** matrix means **linear independence**

Theorem 33. *A basis is a minimal spanning set of V; that is the elements of the basis span V but you cannot delete any of these elements and still get all of V.*

Basis and Dim of four subspaces:

Rank [r] : Number of pivots matrix has

Let A be an $m \times n$ matrix with rank r

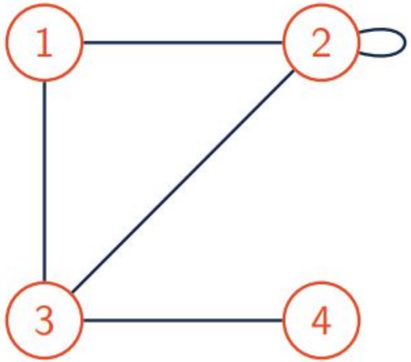
- dim Nul(A) = n - r
- dim Col(A) = r
- dim Nul(A^T) = m - r
- dim Col(A^T) = r

Graphs and Adjacency Matrices

A **graph** is a set of nodes (or: vertices) that are connected through edges.

Definition. Let $\mathcal{G}$ be a graph with n nodes. The **adjacency matrix** of $\mathcal{G}$ is the $n \times n$-matrix $A = (a_{ij})$ such that

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge between node } i \text{ and node } j \\ 0 & \text{otherwise .} \end{cases}$$



$$\begin{array}{cccc} N1 & N2 & N3 & N4 \end{array}$$

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Node 1: Connected to N2 & N3

Node 2: Connected to N1, N2, & N3

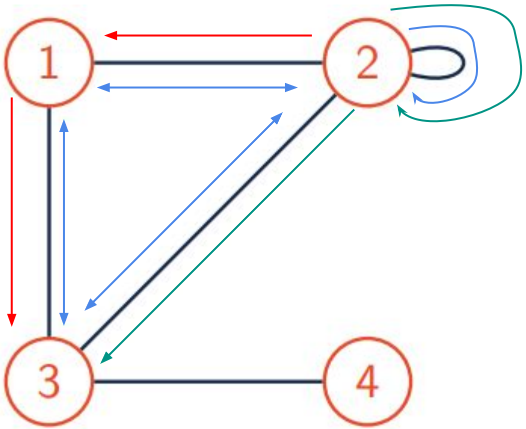
Node 3: Connected to N1, N2 & N4

Node 4: Connected to N3

Walks and Paths

Definition. A **walk of length** k on a graph of is a sequence of $k+1$ vertices and k edges between two nodes (including the start and end) that may repeat.A **path** is walk in which all vertices are distinct.

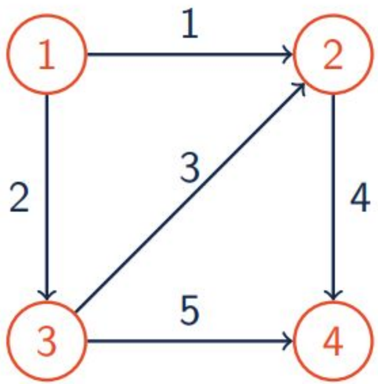
Example. Count the number of walks of length 2 from node 2 to node 3 and the number of walks of length 3 from node 3 back to node 3:



- Node 2 to Node 3: 2 walks of length 2
- Node 3 to Node 3: 3 walks of length 3

Directed Graphs

Definition. A **directed graph** is a set of vertices connected by edges, where the edges have a direction associated with them.



N1 N2 N3 N4

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Node 1: Nothing pointing to N1

Node 2: N1 and N3 pointing to N2

Node 3: N1 points to N3

Node 4: N2 and N3 pointing to N4

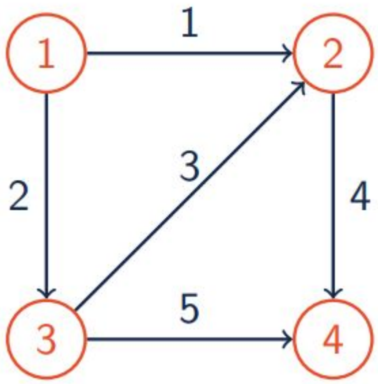
Definition. Let G be a directed graph with m edges and n nodes. The **adjacency matrix** of G is the $n \times n$ matrix $A = (a_{i,j})_{i,j}$ with

$$a_{i,j} = \begin{cases} 1, & \text{if there is a directed edge from node } j \text{ to node } i \\ 0, & \text{otherwise} \end{cases}$$

Edge–Node Incidence

Definition. Let G be a directed graph with m edges and n nodes. The **edge-node incidence matrix** of G is the $m \times n$ matrix $A = (a_{i,j})_{i,j}$ with

$$a_{i,j} = \begin{cases} -1, & \text{if edge } i \text{ leaves node } j \\ +1, & \text{if edge } i \text{ enters node } j \\ 0, & \text{otherwise} \end{cases}$$



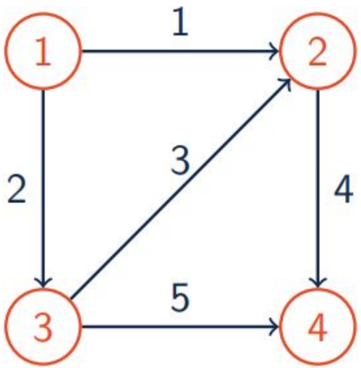
$$\begin{array}{cccc} N1 & N2 & N3 & N4 \end{array}$$
$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Edge 1: Leaves N1; Enters N2
Edge 2: Leaves N1; Enters N3
Edge 3: Leaves N3; Enters N2
Edge 4: Leaves N2; Enters N4
Edge 5: Leaves N3; Enters N4

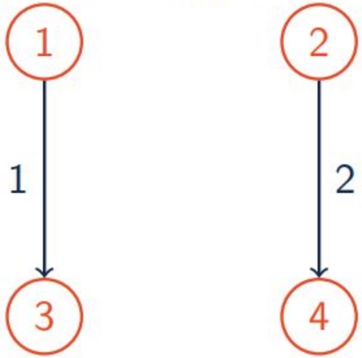
‘Connectedness’

Definition. A **connected component** of an undirected graph is a part in which any two vertices are connected to each other by paths, and which is connected to no additional vertices in the rest of the graph. The connected components of a directed graph are those of its underlying undirected graph. A graph is **connected** if only has one connected component.

A graph with one connected component:



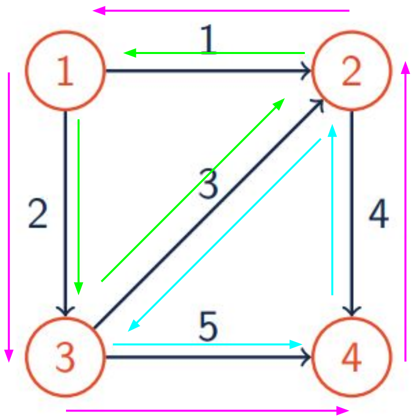
A graph with two connected components:



Theorem 40. *Let $\mathcal{G}$ be a directed graph and let A be its edge-node incidence matrix. Then* $\dim \mathrm{Nul}(A)$ *is equal to the number of connected components of* $\mathcal{G}$.

Cycles

Definition. A **cycle** in an undirected graph is a path in which all edges are distinct and the only repeated vertices are the first and last vertices. By cycles of a directed graph we mean those of its underlying undirected graph.



Cycle 1 + Cycle 2 = Cycle 3

$$\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Theorem 41. *Let $\mathcal{G}$ be a directed graph and let A be its edge-node incidence matrix. Then the cycle space of $\mathcal{G}$ is equal to* $\mathrm{Nul}(A^T)$.

Orthogonal Complements

Definition. Let W be a subspace of $\mathbb{R}^n$. The **orthogonal complement** of W is the subspace $W^\perp$ of all vectors that are orthogonal to W; that is

$$W^\perp := \{\mathbf{v} \in \mathbb{R}^n \ : \ \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\}.$$

Some helpful theorems:

- $(W^\perp)^\perp = W$
- $\mathrm{Nul}(A) = \mathrm{Col}(A^T)^\perp$
- $\mathrm{Nul}(A)^\perp = \mathrm{Col}(A^T)$
- $\mathrm{Nul}(A^T) = \mathrm{Col}(A)^\perp$

Theorem 43. *Let V be a subspace of $\mathbb{R}^n$. Then $\dim V + \dim V^\perp = n$.*

Coordinates

Standard basis (ε):

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Generally, if $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p$ are a basis B of vector space V, the coordinate vector of any vector **w** in V is:

$$\mathbf{w}_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_p \end{bmatrix}, \qquad \text{if } \mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

This coordinate vector is unique!

Coordinate Example

Let $V = \mathbb{R}^2$, and consider the bases

$$\mathcal{B} := \left(\mathbf{b}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix}1\\-1\end{bmatrix}\right)$$

$$\mathcal{E} := \left(\mathbf{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}\right)$$

$\mathbf{w} = \begin{bmatrix}3\\-1\end{bmatrix}$. Determine $\mathbf{w}_\mathcal{B}$ and $\mathbf{w}_\mathcal{E}$

We want to find '**w**' in terms of **B**'s and **E**'s coordinate planes

$$\begin{bmatrix}3\\-1\end{bmatrix} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} = 1\begin{bmatrix}1\\1\end{bmatrix} + 2\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\mathbf{w}_\mathcal{B} = \begin{bmatrix}1\\2\end{bmatrix}$$

This is what [3,-1] looks like in 'basis' **B**

$$\begin{bmatrix}3\\-1\end{bmatrix} = c_1\begin{bmatrix}1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\end{bmatrix} = 3\begin{bmatrix}1\\0\end{bmatrix} + (-1)\begin{bmatrix}0\\1\end{bmatrix}$$

$$\mathbf{w}_\mathcal{E} = \begin{bmatrix}3\\-1\end{bmatrix}$$

This is what [3,-1] looks like in 'basis' **E**

Change of Basis Matrix

Definition. Let $\mathcal{B}$ and $\mathcal{C}$ be two bases of $\mathbb{R}^n$. The **change of basis matrix** $I_{\mathcal{C},\mathcal{B}}$ is the matrix such that for all $\mathbf{v} \in \mathbb{R}^n$

$$I_{\mathcal{C},\mathcal{B}}\mathbf{v}_{\mathcal{B}} = \mathbf{v}_{\mathcal{C}}$$

Matrix allowing us to go from coordinates **mapped in B** to be **mapped onto C**

Theorem 45. *Let $\mathcal{B} = (\mathbf{b}_1, \ldots, \mathbf{b_n})$ be a basis of $\mathbb{R}^n$. Then*

$$I_{\mathcal{E}_n,\mathcal{B}} = \begin{bmatrix} \mathbf{b}_1 & \ldots & \mathbf{b}_n \end{bmatrix}$$

That is, for all $\mathbf{v} \in \mathbb{R}^n$,

$$\mathbf{v} = \begin{bmatrix} \mathbf{b}_1 & \ldots & \mathbf{b}_n \end{bmatrix} \mathbf{v}_{\mathcal{B}}.$$

How do we compute change of basis matrix:

$$I_{\mathcal{B},\mathcal{C}}?$$

What we know:

- $I_{En,B}$ = Matrix that maps **coordinates in B** onto **Standard**
- $I_{B,En}$ = Matrix that maps **coordinates in Standard** onto **B**

- $I_{En,C}$ = Matrix that maps **coordinates in C** onto **Standard**
- $I_{C,En}$ = Matrix that maps **coordinates in Standard** onto **C**

$$I_{\mathcal{B},\mathcal{E}_n} I_{\mathcal{E}_n,\mathcal{C}}$$

From right to left:
We map coordinates **from C into the standard** coordinate plane, **then**, we map the newly acquired **standard coordinates onto B's coordinate plane**

AKA: $I_{\mathcal{B},\mathcal{C}}$

Orthogonal and Orthonormal Bases

Definition. An **orthogonal basis** (an **orthonormal basis**) is an orthogonal set of vectors (an orthonormal set of vectors) that forms a basis.

Theorem 47. *Let $\mathcal{B} := (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ be an orthogonal basis of $\mathbb{R}^n$, and let $\mathbf{v} \in \mathbb{R}^n$. Then*

$$\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{b}_1}{\mathbf{b}_1 \cdot \mathbf{b}_1}\mathbf{b}_1 + \ldots + \frac{\mathbf{v} \cdot \mathbf{b}_n}{\mathbf{b}_n \cdot \mathbf{b}_n}\mathbf{b}_n.$$

When $\mathcal{B}$ is orthonormal, then $\mathbf{b}_i \cdot \mathbf{b}_i = 1$ for $i = 1, \ldots, n$.

Theorem 48. *Let $\mathcal{U} = (\mathbf{u}_1, \ldots, \mathbf{u_n})$ be an orthonormal basis of $\mathbb{R}^n$. Then*

$$I_{\mathcal{U},\mathcal{E}_n} = \begin{bmatrix} \mathbf{u}_1 & \ldots & \mathbf{u}_n \end{bmatrix}^T.$$

Why? An $n \times n$-matrix Q is **orthogonal** if $Q^{-1} = Q^T$

Linear Transformation

Definition. Let V and W be vector spaces. A map $T : V \to W$ is a **linear transformation** if

$$T(a\mathbf{v} + b\mathbf{w}) = aT(\mathbf{v}) + bT(\mathbf{w})$$

for all $\mathbf{v}, \mathbf{w} \in V$ and all $a, b \in \mathbb{R}$.

$$T(\mathbf{0}_V) = T(0 \cdot \mathbf{0}_V) = 0 \cdot T(\mathbf{0}_V) = \mathbf{0}_W \rightsquigarrow T(\mathbf{0}_V) = \mathbf{0}_W$$

Check linearity with the zero vector!

Theorem 50. *Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there is a $m \times n$ matrix A such that*

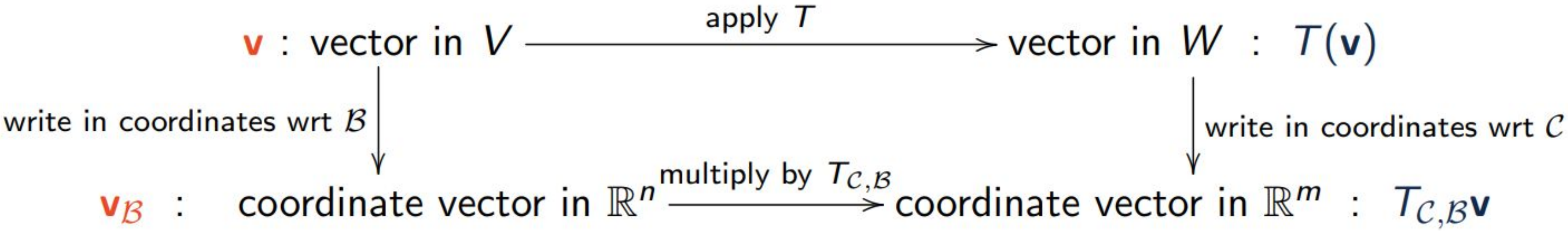
- $T(\mathbf{v}) = A\mathbf{v}$, *for all* $\mathbf{v} \in \mathbb{R}^n$.
- $A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \end{bmatrix}$, *where* $(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)$ *is the standard basis of* $\mathbb{R}^n$.

Remark. We call this A the **coordinate matrix of** T with respect to the standard bases - we write $T_{\mathcal{E}_m,\mathcal{E}_n}$.

Coordinate matrices

Theorem 51. *Let V, W be two vector space, let $\mathcal{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ be a basis of V and $\mathcal{C} = (\mathbf{c}_1, \ldots, \mathbf{c}_m)$ be a basis of W, and let $T\colon V \to W$ be a linear transformation. Then there is a $m \times n$ matrix $T_{\mathcal{C},\mathcal{B}}$ such that*

- $T(\mathbf{v})_{\mathcal{C}} = T_{\mathcal{C},\mathcal{B}}\mathbf{v}_{\mathcal{B}}, \quad$ *for all* $\mathbf{v} \in V$.
- $T_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} T(\mathbf{b}_1)_{\mathcal{C}} & T(\mathbf{b}_2)_{\mathcal{C}} & \ldots & T(\mathbf{b}_n)_{\mathcal{C}} \end{bmatrix}$.



Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be such that $T(\mathbf{v}) = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \mathbf{v}$.

$$\mathcal{B} := \left(\mathbf{b}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

Compute $T_{\mathcal{B},\mathcal{B}}$. Let $\mathbf{v} = \mathbf{b}_1 + \mathbf{b}_2$. Use $T_{\mathcal{B},\mathcal{B}}$ to compute $T(\mathbf{v})$

$$\mathbf{v}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$T_{\mathcal{B},\mathcal{B}} = \begin{bmatrix} T(\mathbf{b}_1)_{\mathcal{B}} & T(\mathbf{b}_2)_{\mathcal{B}} \end{bmatrix}$$

$$T(\mathbf{v})_{\mathcal{B}} = T_{\mathcal{B},\mathcal{B}} \mathbf{v}_{\mathcal{B}}$$

$$T(\mathbf{b}_1) = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2\mathbf{b}_1 + 0\mathbf{b}_2$$

$$T_{\mathcal{B},\mathcal{B}} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$T(\mathbf{b}_2) = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0\mathbf{b}_1 + 4\mathbf{b}_2$$

$$T(\mathbf{v}) = 2\mathbf{b}_1 + 4\mathbf{b}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

Theorem 52. *Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation and $\mathcal{A}$ and $\mathcal{B}$ be two bases of $\mathbb{R}^m$ and $\mathcal{C}, \mathcal{D}$ be two bases of $\mathbb{R}^n$. Then*

$$T_{\mathcal{C},\mathcal{A}} = I_{\mathcal{C},\mathcal{D}} T_{\mathcal{D},\mathcal{B}} I_{\mathcal{B},\mathcal{A}}.$$

Online Resources

Vector Spaces:
https://www.youtube.com/watch?v=XDvSsDsLVLs&ab_channel=TrevTutor

Linear Combinations, Spans, and Basis Vectors:
https://www.youtube.com/watch?v=k7RM-ot2NWY&list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab&index=2&ab_channel=3Blue1Brown

Change of Basis:
https://www.youtube.com/watch?v=P2LTAUO1TdA&list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab&index=13&ab_channel=3Blue1Brown

Abstract Vector Spaces:
https://www.youtube.com/watch?v=TgKwz5Ikpc8&list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab&index=16&ab_channel=3Blue1Brown

In-Person Resources

CARE Drop-in tutoring:
7 days a week on the 4th floor of Grainger Library!
Sunday - Thursday 12pm-10pm
Friday & Saturday 12-6pm

Course Office hours:
TAs: Mondays - Thursdays 5-7pm
Mondays: Loomis 136
Tuesdays: Transportation Building 203
Wednesdays: Loomis 136
Thursdays: Loomis 139
Instructors: Chuang (PL1): M 3-5PM in CAB 233
Luecke (PL2,PL3): Tu 3:30-5:30 in Altgeld 105

Subject	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
Math 257	1pm-5pm 6pm-9pm	2pm-10pm	12pm-2pm 3pm-10pm	12pm-2pm 4pm-6pm 7pm-10pm	1pm-7pm 8pm-10pm	1pm-5pm	12pm-2pm 3pm-5pm

Questions?

Join the queue to see the worksheet and slides!