Center for Academic Resources in Engineering (CARE)
Peer Exam Review Session

MATH 257 – Linear Algebra with Computational Applications

Midterm 2 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: October 16th, 6:30-8PM Alice, Carlos, JD

Session 2: October 17th, 7-8:30PM Danielle, Rishi

Can't make it to a session? Here's our schedule by course:

https://care.grainger.illinois.edu/tutoring/schedule-by-subject

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/955
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. Given

$$A = \begin{bmatrix} 4 & -3 & 1 \\ -8 & 14 & -1 \\ 4 & 13 & 4 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} 3 \\ 22 \\ 63 \end{bmatrix}$$

a) Compute REF(A) without swapping any rows. Write down all row operations used in the process

$$\begin{bmatrix} 4 & -3 & 1 \\ -8 & 14 & -1 \\ 4 & 13 & 4 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 2R_1} \begin{bmatrix} 4 & -3 & 1 \\ 0 & 8 & 1 \\ 4 & 13 & 4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 4 & -3 & 1 \\ 0 & 8 & 1 \\ 0 & 16 & 3 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 4 & -3 & 1 \\ 0 & 8 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

b) Express REF(A) as the product of elementary matrices and A (Hint: Use your process from a)

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{R_3 \to R_3 - 2R_2} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{R_3 \to R_3 - R_1} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_2 \to R_2 + 2R_1} \underbrace{\begin{bmatrix} 4 & -3 & 1 \\ -8 & 14 & -1 \\ 4 & 13 & 4 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 4 & -3 & 1 \\ 0 & 8 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{REF(A) = U}$$

c) Express A as the product of a lower triangular matrix and an upper triangular matrix A = LU

$$E_3 E_2 E_1 A = U$$
$$A = E_1^{-1} E_2^{-1} E_3^{-1} U$$

$$\begin{bmatrix} 4 & -3 & 1 \\ -8 & 14 & -1 \\ 4 & 13 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & -3 & 1 \\ 0 & 8 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 4 & -3 & 1 \\ 0 & 8 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

d) Use the LU decomposition of A to solve the linear system $A\boldsymbol{x} = \boldsymbol{b}$

$$A\boldsymbol{x} = \boldsymbol{b}$$
$$(LU)\boldsymbol{x} = \boldsymbol{b} \to \begin{cases} L\boldsymbol{y} = \boldsymbol{b} \\ U\boldsymbol{x} = \boldsymbol{y} \end{cases}$$

Solve $L\boldsymbol{y} = \boldsymbol{b}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \boldsymbol{y} = \begin{bmatrix} 3 \\ 22 \\ 63 \end{bmatrix} \to \quad \boldsymbol{y} = \begin{bmatrix} 3 \\ 28 \\ 4 \end{bmatrix}$$

Solve $U\boldsymbol{x} = \boldsymbol{y}$:

$$\begin{bmatrix} 4 & -3 & 1 \\ 0 & 8 & 1 \\ 0 & 0 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 3 \\ 28 \\ 4 \end{bmatrix} \to \quad \vec{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

2. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be defined:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 4 & -1 & 2 & 4 \\ 5 & 4 & 0 & 5 & 5 \\ 8 & 4 & 2 & 6 & 2 \end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For the following statements, determine whether they are true or false. If they are false, explain why.

a) Null($\boldsymbol{A}$) is a subspace of $\mathbb{R}^5$

Vectors in the nullspace of $\boldsymbol{A}$ must result in the zero vector when multiplied with $\boldsymbol{A}$. The only dimensions that make sense are a 5 x 1 column vector. Thus, the nullspace of $\boldsymbol{A}$ must be a subspace of $\mathbb{R}^5$ and this statement is $\boxed{True}$.

b) dim Col($\boldsymbol{B}$) = dim Null($\boldsymbol{B}$)

Observe that rows 2 and 3 are scalar multiples of each other, but are linearly independent from row 1. Row 4 is all zeros, so it is always linearly dependent. This means $\boldsymbol{B}$ has two pivot variables and two free variables. (You can also reduce down to RREF to check this!)

dim Col($\boldsymbol{B}$) = the number of pivot variables = 2, while dim Null($\boldsymbol{B}$) = the number of free variables = 2. This statement is therefore $\boxed{True}$.

c) The vector $\begin{bmatrix} -1 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}$ spans the nullspace of $\boldsymbol{A}$, and is a solution to the vector equation A$\boldsymbol{x} = \mathbf{0}$

Check the statement by multiplying the matrix $\boldsymbol{A}$ with the given vector:

$$\begin{bmatrix} 1 & 4 & -1 & 2 & 4 \\ 5 & 4 & 0 & 5 & 5 \\ 8 & 4 & 2 & 6 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Next, check that the vector spans the nullspace by reducing A to RREF:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -3 \end{bmatrix}$$

Since the RREF of $\boldsymbol{A}$ has two free variables, dim Null($\boldsymbol{A}$) = 2. To span the nullspace, you must have the whole basis for the nullspace, but we only have one vector. One vector can never span a nullspace with a dimension of 2. Thus, this statement is $\boxed{False}$.

d) If the row operation $R_4 \to R_4 + R_1$ is performed on matrix $\boldsymbol{B}$, the column space of the resulting matrix is the same as Col($\boldsymbol{B}$)

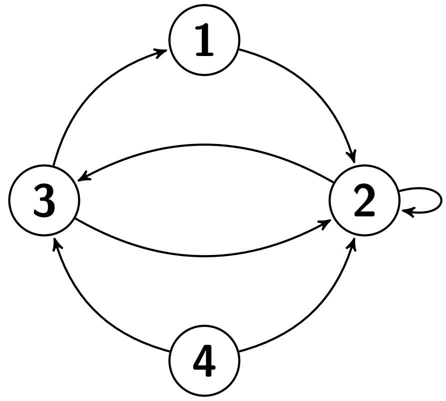
Observe that column 4 = column 3 + 2*column 1. Thus, the column space of $\boldsymbol{B}$ can be spanned by the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$. However, after performing the row operation, this is the resulting matrix:

$$\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 1 & 0 & 1 & 3 \end{bmatrix}$$

Now, the column space is spanned by the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$. These are not the same space!

This statement is $\boxed{False}$.

3. What is the adjacency matrix for the following graph?



$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad A_{i,j} \begin{cases} 1, & \text{if vertex i has directed edge to vertex j} \\ 0, & \text{otherwise} \end{cases}$$

4. For the set S $= \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$, state what is necessary for it to be a basis of $\mathbb{R}^3$

All of the vectors in the set S must be linearly independent and span(S) $= \mathbb{R}^3$

a) For the same set S, state what is necessary to make it a basis of $\mathbb{R}^2$

There must be a subset, $U \subseteq S$ consisting of two vectors such that span(U) $= \mathbb{R}^2$. In other words, two of the three vectors in S must be linearly dependent.

5. The vectors

$$\boldsymbol{u_1} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \boldsymbol{u_2} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}, \boldsymbol{u_3} = \begin{bmatrix} -8 \\ 12 \\ -4 \end{bmatrix}, \boldsymbol{u_4} = \begin{bmatrix} 1 \\ 37 \\ -17 \end{bmatrix}, \boldsymbol{u_5} = \begin{bmatrix} -3 \\ -5 \\ 8 \end{bmatrix}$$

span $\mathbb{R}^3$. Find the minimally spanning subset of set A $= \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}, \boldsymbol{u_5}\}$.
Hint: the minimally spanning subset of $\mathbb{R}^3$ should have three vectors.

We can create a matrix using our vectors and bring it to RREF

$$\begin{bmatrix} 2 & 1 & -8 & 1 & -3 \\ -3 & 4 & 12 & 37 & -5 \\ 1 & -2 & -4 & -17 & 8 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -4 & -3 & 0 \\ 0 & 1 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

With $\boldsymbol{u_3}$ and $\boldsymbol{u_4}$ the set is linearly dependent, thus we remove them to make a new set $\widetilde{A}$, only considering $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_5}$.

Let $\widetilde{A} \subseteq A$, where $\widetilde{A} = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_5}\}$. Then the solution $a\boldsymbol{u_1} + b\boldsymbol{u_2} + c\boldsymbol{u_5}$ is

$$\left(\begin{array}{ccc|c} 2 & 1 & -3 & 0 \\ -3 & 4 & -5 & 0 \\ 1 & -2 & 8 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

The only solution is $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, which is trivial. Thus, $\widetilde{A}$ is linearly independent and since $\dim(\widetilde{A}) = 3$, $\text{span}(\widetilde{A}) = \mathbb{R}^3$. Therfore $\widetilde{A}$ is a basis of $\mathbb{R}^3$

6. Let $S \subseteq M_{3\times 3}(\mathbb{R})$ be all 3×3 symmetric matrices of real entries. Find its dimension.

Any $A \in S$ is in the form $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$, thus dim S = 6

a) Find a basis of A. (Hint: Any matrix $A \in S$ is symmetric)

$$\text{Basis} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}$$

7. For the following statements, state whether they are true or false and why.

a) True/False : A vector space cannot have more than one basis

$\boxed{False}$. Basis is not unique, you can have multiple forms of a basis.

b) True/False : If a vector space has a finite basis, then the number of vectors in each minimally spanning basis is the same.

$\boxed{True}$. If two presumed minimally spanning bases have different number of vectors, then either one of them is linearly dependent or or one of them doesn't span the vector space.

c) True/False : If S generates V (span(S) = V), then every vector in V can be written as a linear combination of vectors in S in only one way.

$\boxed{False}$. "S generates V" does not imply that S is a basis of V. For example, $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$, span(S) $= \mathbb{R}^2$, yet the linear equation $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ has infinite solutions (i.e. there are free variables)

8. Let A be defined as $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$. Compute A^{-1} and A^T. What type of matrix is A?

$$A^{-1} = A^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \text{, A is orthogonal}$$

9. Consider the following basis: $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \\ -3 \end{bmatrix}. \begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix} \right\}$ and standard basis $\mathcal{E}$

and $\quad \boldsymbol{V_{\mathcal{E}}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \boldsymbol{W_{\mathcal{E}}} = \begin{bmatrix} -48 \\ 26 \\ 48 \end{bmatrix} \quad \boldsymbol{U_{\mathcal{E}}} = \begin{bmatrix} -23 \\ 16 \\ 26 \end{bmatrix} \quad \boldsymbol{X_{\mathcal{B}}} = \begin{bmatrix} 0 \\ \frac{1}{9} \\ 0 \end{bmatrix} \quad \boldsymbol{Y_{\mathcal{B}}} = \begin{bmatrix} -3 \\ -2 \\ \frac{1}{2} \end{bmatrix}$

Compute $\boldsymbol{V_{\mathcal{B}}}$, $\boldsymbol{W_{\mathcal{B}}}$, $\boldsymbol{U_{\mathcal{B}}}$, $\boldsymbol{X_{\mathcal{E}}}$, $\boldsymbol{Y_{\mathcal{E}}}$

From the given information we see,

$$I_{\mathcal{E},\mathcal{B}} = \begin{bmatrix} 1 & -3 & -4 \\ 2 & -3 & 2 \\ 2 & -3 & 4 \end{bmatrix} \text{, thus it follows } I_{\mathcal{B},\mathcal{E}} = I_{\mathcal{E},\mathcal{B}}^{-1} = \begin{bmatrix} -1 & 4 & -3 \\ -\frac{2}{3} & 2 & -\frac{5}{3} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\boldsymbol{V_{\mathcal{B}}} = I_{\mathcal{B},\mathcal{E}}\boldsymbol{V_{\mathcal{E}}} = \begin{bmatrix} 4 \\ 2 \\ \frac{1}{2} \end{bmatrix} \quad \boldsymbol{W_{\mathcal{B}}} = I_{\mathcal{B},\mathcal{E}}\boldsymbol{W_{\mathcal{E}}} = \begin{bmatrix} 8 \\ 4 \\ 11 \end{bmatrix} \quad \boldsymbol{U_{\mathcal{B}}} = I_{\mathcal{B},\mathcal{E}}\boldsymbol{U_{\mathcal{E}}} = \begin{bmatrix} 9 \\ 4 \\ 5 \end{bmatrix}$$

$$\boldsymbol{X_{\mathcal{E}}} = I_{\mathcal{E},\mathcal{B}}\boldsymbol{X_{\mathcal{B}}} = \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} \quad \boldsymbol{Y_{\mathcal{E}}} = I_{\mathcal{E},\mathcal{B}}\boldsymbol{Y_{\mathcal{B}}} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$