The Grainger College of Engineering
Center for Academic Resources in Engineering

MATH 241

Midterm 1 Review

Keep in mind that this presentation was created by CARE tutors, and while it is thorough, it is not comprehensive.

QR Code to the Queue

The queue contains the worksheet and the solution to this review session

Dot Product

Dot product from components. #rvv-es

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Dot product from length/angle. #rvv-ed

$$\vec{a} \cdot \vec{b} = ab \cos\theta$$

Length and angle from dot product. #rvv-el

$$a = \sqrt{\vec{a} \cdot \vec{a}}$$
$$\cos\theta = \frac{\vec{b} \cdot \vec{a}}{ba}$$

Cross Product

Cross product in components. #rvv-ex

$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2)\,\hat{\imath} + (a_3 b_1 - a_1 b_3)\,\hat{\jmath} + (a_1 b_2 - a_2 b_1)\,\hat{k}$$

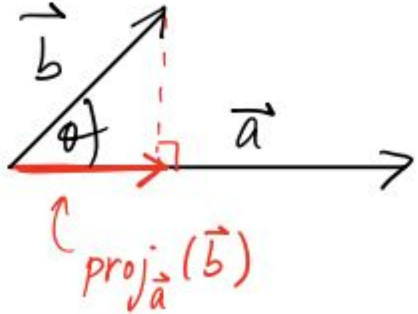
Cross products of basis vectors. #rvv-eo

$$\hat{\imath} \times \hat{\jmath} = \hat{k} \qquad \hat{\jmath} \times \hat{k} = \hat{\imath} \qquad \hat{k} \times \hat{\imath} = \hat{\jmath}$$

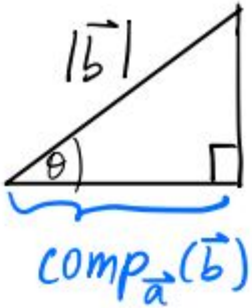
$$\hat{\jmath} \times \hat{\imath} = -\hat{k} \qquad \hat{k} \times \hat{\jmath} = -\hat{\imath} \qquad \hat{\imath} \times \hat{k} = -\hat{\jmath}$$

Projection and Components

$$\text{proj}_{\vec{a}}(\vec{b}) = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\frac{\vec{a}}{|\vec{a}|}$$



$$\text{comp}_{\vec{a}}(\vec{b}) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

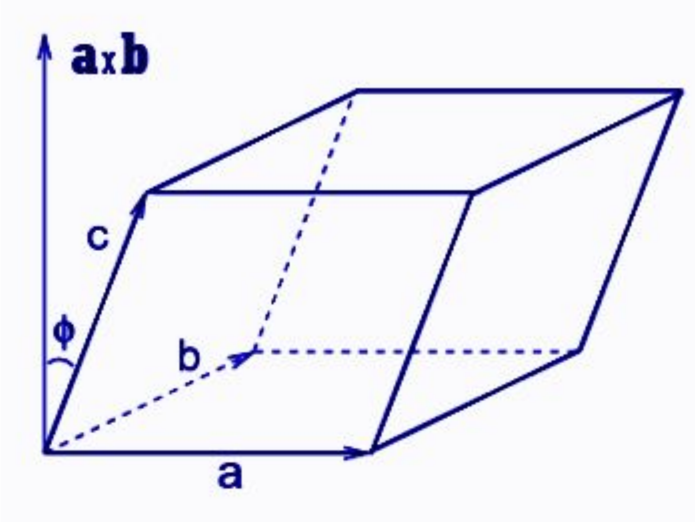


Scalar Triple Product

- A · (B x C)
- Represents the parallelepiped volume enclosed by the three vectors

$$\vec{A} = \langle a_1, a_2, a_3 \rangle, \;\; \vec{B} = \langle b_1, b_2, b_3 \rangle, \;\; \vec{C} = \langle c_1, c_2, c_3 \rangle$$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$



Equations for Lines and Planes

$$Ax + By + Cz = D$$

- Describes a plane in which *A*, *B*, and *C* are the components of the normal vector
- To find *D*, you need a point on the plane:

$$< x_0,\ y_0,\ z_0 >$$

$$D = Ax_0 + By_0 + Cz_0$$

- The equation for a line *L* on a plane can be parametrized:
  - Here, r_0 is a vector between the origin and a point on the plane
  - And *v* is a line on the plane

$$L = \vec{r_0} + t\vec{v}$$

$$\vec{r_0} = < x_0,\ y_0,\ z_0 >$$

$$\vec{v} = < v_1, v_2, v_3 >$$

$$\left\{ \begin{array}{rcl} x(t) & = & x_0 + tv_1 \\ y(t) & = & y_0 + tv_2 \\ z(t) & = & z_0 + tv_3 \end{array} \right\}$$

Example Question #1

Let **P** be the plane with equation $x + 2z = 0$. Find the distance from the point $(-1, 3, 0)$ to the plane **P**.

Example Solution #1

$$The\ plane\ passes\ through\ (0,0,0)\ and\ the\ normal\ vector\ \vec{N}\ is\ \langle 1,0,2\rangle$$

$$Create\ a\ vector\ \vec{V}\ from\ (0,0,0)\ to\ the\ point\ (-1,3,0) \rightarrow \langle -1,3,0\rangle$$

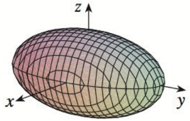
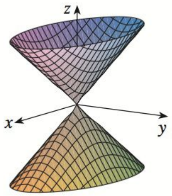
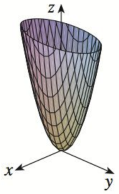
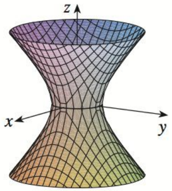
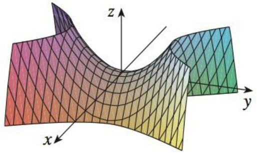
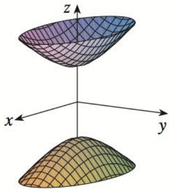
The magnitude of the projection of $\vec{\mathrm{V}}$ onto $\vec{\mathrm{N}}$ will be the distance from the point to the plane

$$proj_{\vec{N}}\vec{V} = \frac{\vec{V}\cdot\vec{N}}{|\vec{N}|^2}\vec{N} = \langle {}^{-1}/_{5}, 0, {}^{-2}/_{5}\rangle$$

$$|proj_{\vec{N}}\vec{V}| = \sqrt{\left({}^{-1}/_{5}\right)^2 + \left({}^{-2}/_{5}\right)^2} = {}^{1}/_{\sqrt{5}}$$

$$The\ distance\ is\ 1/\sqrt{5}$$

Quadric Surface

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

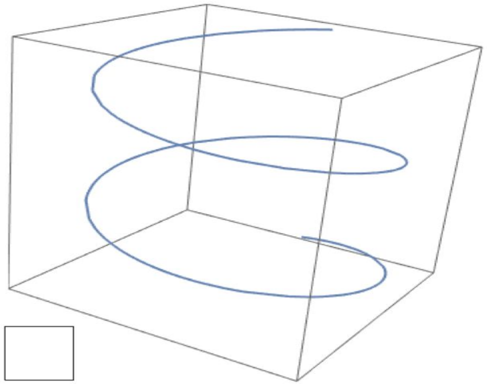
Example Question #2

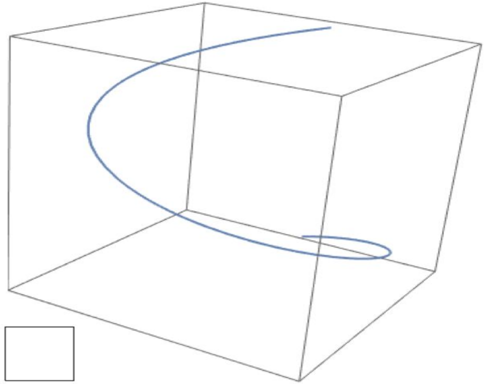
Let C be the curve parameterized by $\mathbf{r}(t) = \langle \sin(t^2), \cos(t^2), t^2 \rangle$ for $0 \leq t \leq 2\sqrt{\pi}$. Check the corresponding picture of C.

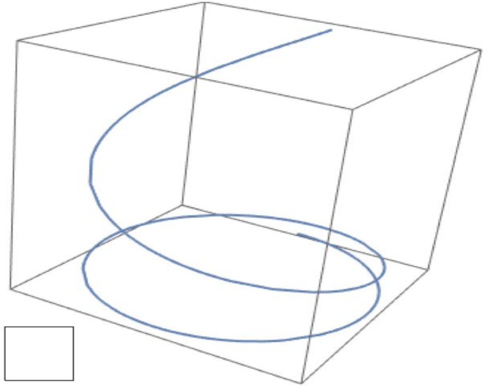
(Pictures are on the next slide)

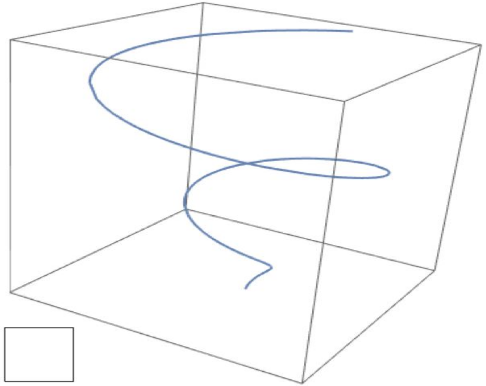
Example Q

Let C be the curve parameterized by $\mathbf{r}(t) = \langle \sin(t^2), \cos(t^2), t^2 \rangle$ for $0 \le t \le 2\sqrt{\pi}$. Check the corresponding picture of C.



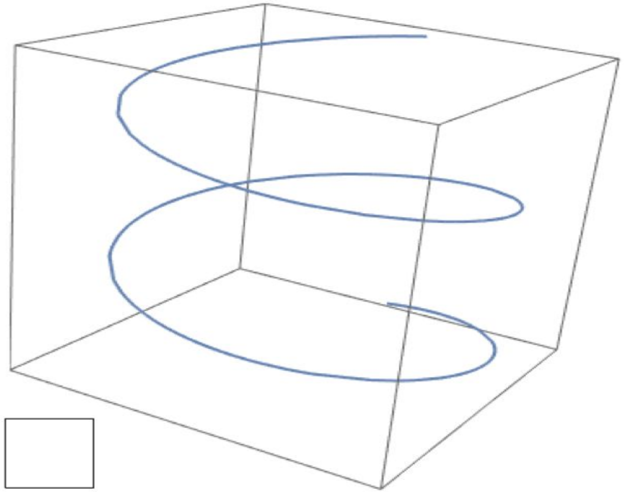






Example Solution #2

$\mathbf{r}(t) = \langle \sin(t^2), \cos(t^2), t^2 \rangle$ for $0 \leq t \leq 2\sqrt{\pi}$



Example Question #3

Find the vector function representing the curve of intersection between the circular cylinder of radius 4 centered on the z-axis and the surface z = xy.

Example Solution #3

Find the vector function representing the curve of intersection between the circular cylinder of radius 4 centered on the z-axis and the surface z = xy.

$$\overrightarrow{r_{cyl}} = \langle 4\cos t, 4\sin t \rangle$$

$$z = xy = 16\cos t \cdot \sin t$$

$$\vec{r}(t) = \langle 4\cos t, 4\sin t, 16\cos t \cdot \sin t \rangle$$

Partial Derivatives

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \qquad f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

$$f(x,y) \quad \Rightarrow \quad f_x(x,y) = \frac{\partial f}{\partial x} \;\&\; f_y(x,y) = \frac{\partial f}{\partial y}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(f_x) = (f_x)_x = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(f_y) = (f_y)_y = f_{yy}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(f_x) = (f_x)_y = f_{xy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(f_y) = (f_y)_x = f_{yx}$$

Linear Approximation

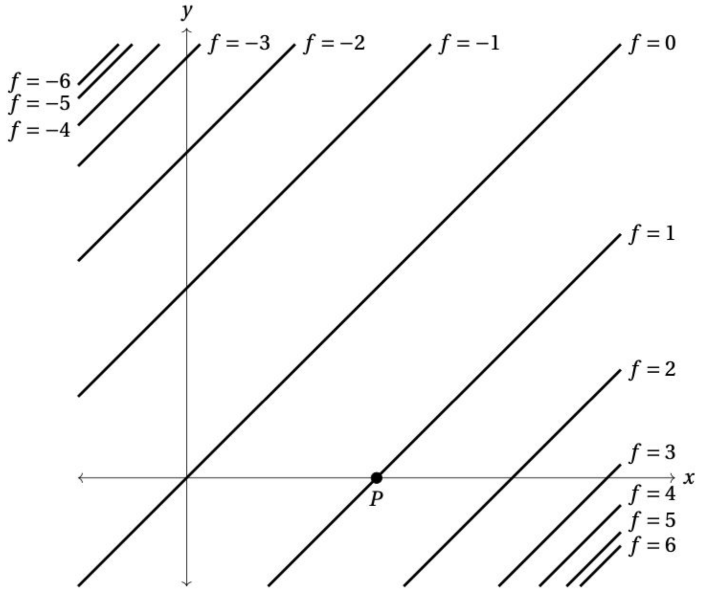
- If z = f(x, y) and f is **differentiable** at (a, b), then the value of f(m, n) can be approximated by

$$f(\mathrm{m},\mathrm{n}) \approx L(\mathrm{m},\mathrm{n})$$

$$L(\mathrm{m},\mathrm{n}) = \mathrm{f}(\mathrm{a},\mathrm{b}) + \mathrm{f_x}(\mathrm{a},\mathrm{b}) \cdot (\mathrm{m}-\mathrm{a}) + \mathrm{f_y}(\mathrm{a},\mathrm{b}) \cdot (\mathrm{n}-\mathrm{b})$$

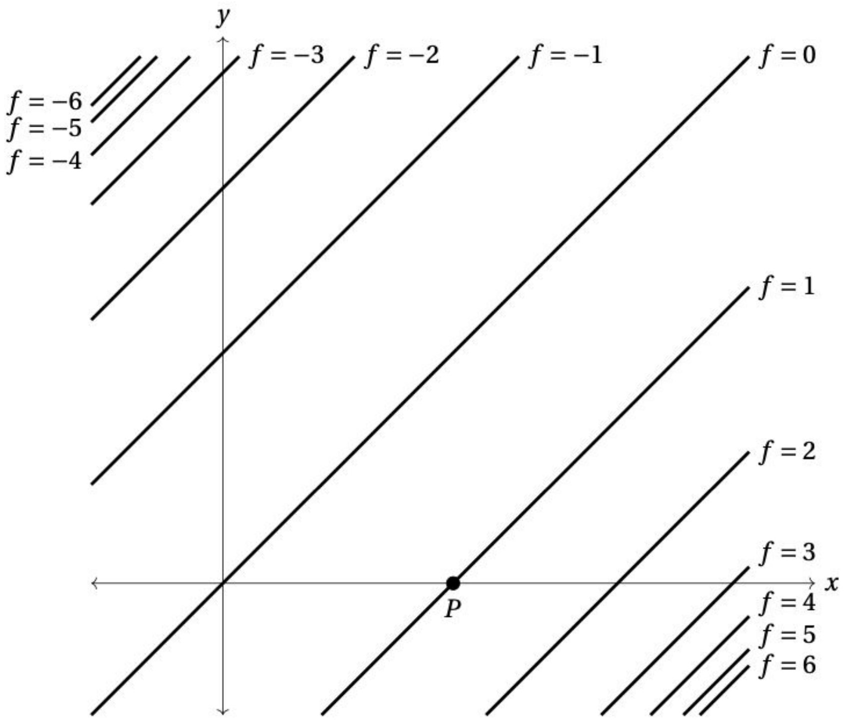
Example Question #1

- A contour map for a function f of x, y, and a point P in the plane are given below. Determine if the following quantities are negative, zero, or positive: $f_x(P)$, $f_{xx}(P)$, $f_{xy}(P)$



Example Solution #1

- $f_x(P)$: positive
- $f_{xx}(P)$: positive
- $f_{xy}(P)$: negative



Limits and Continuity

- When computing multivariable limits,
  - Check **multiple paths** (lines and power functions) to see if there are conflicting values. If so, limits DNE
  - **Factor** (difference of squares)
  - Use **polar coordinates**
  - Try **squeeze theorem**

Example Question #2

- Compute the following limits

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} \qquad \lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8} \qquad \lim_{x\to 0} x^3 \cos\left(\frac{2}{x}\right) \qquad \lim_{(x,y)\to(-1,0)} \frac{x^2+xy+3}{x^2y-5xy+y^2+1}$$

- Determine whether the following function is continuous at (0, 0)

$$f(x,y) = \begin{bmatrix} \dfrac{xy}{x^2+xy+y^2}, & (x,y) \neq (0,0) \\ 0 & ,(x,y) = (0,0) \end{bmatrix}$$

Example Solution #2

- Compute the following limits

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$$ **= 0** (Use polar coordinates)

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$$ **= DNE** (Check $x = y^4$ and $x = -y^4$)

$$\lim_{x\to 0} x^3 \cos\left(\frac{2}{x}\right)$$ **= 0** (Squeeze Theorem)

$$\lim_{(x,y)\to(-1,0)} \frac{x^2+xy+3}{x^2y-5xy+y^2+1}$$ **= 4** (Plug in (–1, 0) directly)

Example Solution #2

- Determine whether the following function is continuous at (0, 0)

$$f(x,y) = \begin{bmatrix} \dfrac{\mathrm{xy}}{\mathrm{x}^2 + \mathrm{xy} + \mathrm{y}^2}, (x,y) \neq (0,0) \\ 0 \qquad\qquad\quad, (x,y) = (0,0) \end{bmatrix}$$

On line y = x, f(x, y) = 1/3 at any point except (0, 0). Since there is a discontinuity at (0, 0), the function is not continuous.

Gradient and Directional Derivatives

$$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

- The gradient will always point **perpendicular to the level curves/surfaces of f**
- $\nabla f = 0$ at a local minimum/maximum

$$D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

- Tells you how the function f changes along the vector u