Center for Academic Resources in Engineering (CARE)
Peer Exam Review Session

Math 241 – Calculus III

Midterm 1 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Sept. 27, 5-6:30 pm Camila, Gabriel

Session 2: Sept. 28, 4-5:30 pm Rose, Pallab

Session 3: Sept. 29, 12 – 1:30 pm Rose, Kewal, Lucas

Can't make it to a session? Here's our schedule by course:

https://care.grainger.illinois.edu/tutoring/schedule-by-subject

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/845
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. Find $\vec{u} \times \vec{v}$ if $\vec{u} = \langle 3, -4, 1\rangle$ and $\vec{v} = \langle 5, 2, -6\rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -4 & 1 \\ 5 & 2 & -6 \end{vmatrix} = \begin{vmatrix} -4 & 1 \\ 2 & -6 \end{vmatrix} \hat{i} - \begin{vmatrix} 3 & 1 \\ 5 & -6 \end{vmatrix} \hat{j} + \begin{vmatrix} 3 & -4 \\ 5 & 2 \end{vmatrix} \hat{k}$$

$$22\hat{i} - (-23)\hat{j} + 26\hat{k}$$

$$\boxed{\langle 22, 23, 26\rangle}$$

2. Find an equation for the plane that passes through the point $P = (1, 2, 3)$ and contains the line L given by the parametric equation:

$x(t) = 1 - 3t$ $\quad$ $y(t) = 3$ $\quad$ $z(t) = 6 + 2t$

for $-\infty < t < \infty$

We always need two pieces of information for the equation of a plane:

- Point on the plane, $P(x_0, y_0, z_0)$
- Vector normal to the plane, $\vec{N}$

The equation of a plane, then, is $0 = \vec{N} \cdot \langle x - x_0,\ y - y_0,\ z - z_0\rangle$

Let $P_0 = (1, 2, 3)$

From the given line equations (when $t = 0$):
$P_1(1, 3, 6)$
Direction vector of the line $\vec{a} = \langle -3, 0, 2\rangle$

In order to compute a cross product and get the normal vector $\vec{N}$, we need one more vector. Subtract points P_1 and P_0. This gives us another vector in the plane that we want between P_1 and P_0.

We get the vector $\vec{b} = \langle 1 - 1, 2 - 3, 3 - 6\rangle = \langle 0, -1, -3\rangle$

Now, $\vec{N} = \vec{a} \times \vec{b} = \langle 2, -9, 3\rangle$

To get the equation of the plane:

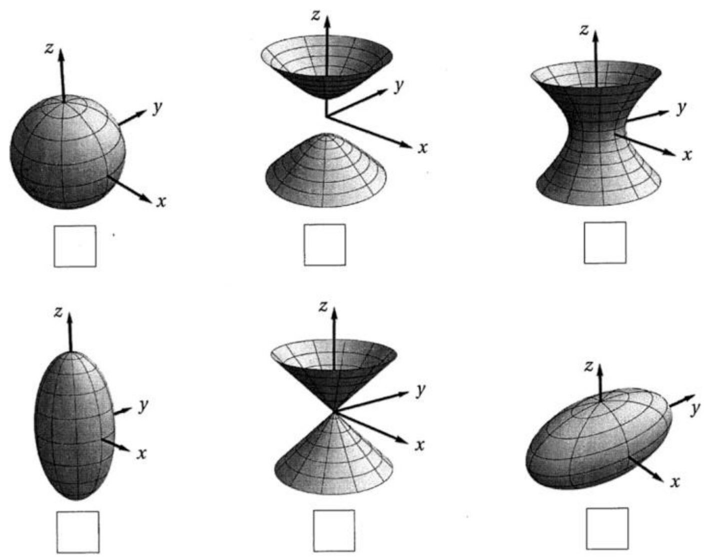
$0 = \vec{N}\ \langle x - x_0, y - y_0, z - z_0\rangle$
$0 = \langle 2, -9, 3\rangle \cdot \langle x - 1, y - 2, z - 3\rangle$

$2x - 9y + 3x = -7$

3. For each equation below, write the corresponding letter in the box next to the picture of the surface it describes

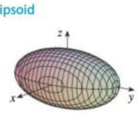
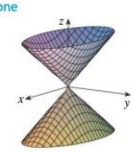
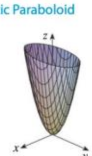
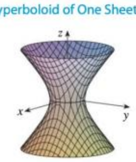
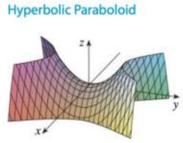
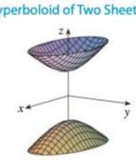
(A) $x^2 + y^2 - z^2 + 1 = 0$

(B) $4x^2 + y^2 + 4z^2 - 1 = 0$



(A) is top middle (hyperboloid of two sheets)
(B) is bottom right (ellipsoid with longest axis along y^)

Here is a little cheat sheet for the equations of quadric surfaces:

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

4. Construct an equation for a plane that contains $(2,1,3)$, $(0,-1,0)$, and $(3,2,1)$.
To obtain an equation for a plane, we need to first find the normal vector of the plane. For the given points, we can make two vectors out of the three points with a common tail. The cross product of these two vectors will then be the normal vector of the plane. Let's assign the points with a letter: $A = (2,1,3), B = (0,-1,0), C = (3,2,1)$.

$$\vec{BA} = (2,1,3) - (0,-1,0) = \langle 2,2,3\rangle$$

$$\vec{BC} = (3,2,1) - (0,-1,0) = \langle 3,3,1\rangle$$

$$\vec{n} = \vec{BA} \times \vec{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 3 \\ 3 & 3 & 1 \end{vmatrix} = \langle -7,7,0\rangle$$

The plane equation is now $-7x + 7y + d = 0$. To solve for d, we can plug in any of the given points. In this case, we choose (2,1,3).

$$-14 + 7 + d = 0 \rightarrow d = 7$$

$$\boxed{-7x + 7y + 7 = 0}$$

5. Consider two points A and B.

$$A = (0, 7, 2)$$
$$B = (1, 2, 0)$$

(a) Find the vector that represents the displacement between the points (vector drawn from A to B)

(b) What is the projection of this vector onto $\vec{r} = \langle 5, -2, 7 \rangle$?

(c) What is the projection of the vector from part (a) onto the plane represented by the equation $5x - 2y + 7z = 10$

(a)

$$\vec{s} = B - A = \boxed{\langle 1, -5, -2 \rangle}$$

(b) To project onto $\vec{r} = \langle 5, -2, 7 \rangle$ we use the projection formula

$$\text{proj}_{\vec{r}}\, \vec{s} = \frac{\vec{r} \cdot \vec{s}}{\vec{r} \cdot \vec{r}}\, \vec{r}$$

$$\boxed{\left\langle \frac{5}{78}, \frac{-2}{78}, \frac{7}{78} \right\rangle}$$

(c) Notice that the normal vector of the plane is the same as vector $\vec{r}$ in part (b). To project $\vec{s}$ onto the plane, we can do an orthogonal projection of $\vec{s}$ onto the plane's normal vector $\vec{r}$ and then do vector subtraction to find the orthogonal component (which is what we want).

$$\text{orth}_{\vec{r}}\, \vec{s} = \vec{s} - \text{proj}_{\vec{r}}(\vec{s}) = \langle 1, -5, -2 \rangle - \left\langle \frac{5}{78}, \frac{-2}{78}, \frac{7}{78} \right\rangle$$

$$\boxed{\left\langle \frac{73}{78}, -\frac{194}{39}, -\frac{163}{78} \right\rangle}$$

6. Determine if the two lines intersect. If so, find their point of intersection.

$$\begin{cases} x = 3t - 3 \\ y = -2t + 1 \\ z = 4t - 2 \end{cases} \qquad \begin{cases} x = 2s + 3 \\ y = 2s - 1 \\ z = s + 2 \end{cases}$$

If these two lines intersect, there will be a value $t = t_0$ and $s = s_0$ (not necessarily the same), such that each component of the first line equals each component of the second line. Thus we can equate the x and y components of each one, solve for t and s, and see if these values also satisfy the z component.

$$3t - 3 = 2s + 3$$
$$-2t + 1 = 2s - 1$$

Solving these two gives $t = \frac{8}{5}$ and $s = -\frac{3}{5}$. We can plug these into the equations for z to see if they will also match up.

$$z = 4\left(\frac{8}{5}\right) - 2 = \frac{22}{5}$$
$$z = -\frac{3}{5} + 2 = \frac{7}{5}$$

These lines do not intersect

7. Find the equation of the line of intersection of the following two planes: $2x + y - 2z = 2$ and $-2x + y + z = 6$.

We are looking for the equation of a line, which means we need a starting point and a direction vector. Finding a starting point is easy, we just need a point that is on both planes. To do this, we will let $z = 0$ and solve the two equations for x and y.

$$2x + y = 2$$
$$-2x + y = 6$$

Solving gives a point on both planes $(-1, 4, 0)$. This will be our starting vector.

Now we need a direction vector. Notice that this vector will be perpendicular to each plane's normal vector. Therefore, we can take a cross product.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ -2 & 1 & 1 \end{vmatrix} = \langle 3, 2, 4 \rangle$$

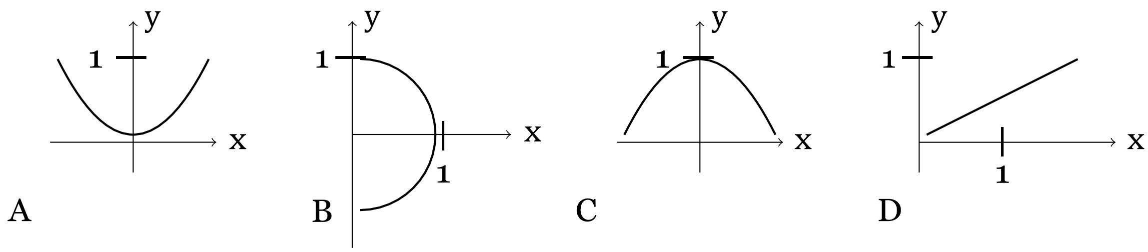
Therefore, the equation of the line of intersection is [1]

[1] Other answers are possible depending on which starting vector you use, but the direction vector should be parallel to this one

$$\boxed{\vec{r}(t) = \langle -1 + 3t, 4 + 2t, 4t\rangle}$$

8. Explain why $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$.
$\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. The dot product of perpendicular vectors is zero.

9. Let $r(t) = \langle \sin(t), \cos^2 t\rangle$, $0 \le t \le 2\pi$. Which graph below represents this curve?



Converting this into a Cartesian equation gives

$$y + x^2 = 1 \to y = 1 - x^2$$

Which is a concave down parabola with its vertex at $(0, 1)$ shown in graph C.

10. Assume you are walking around the surface of a spherical planet with a radius of 2. If you are walking clockwise on the xy-plane, what is the parameterization of the path after circling it twice?

The equation for a sphere (with radius 2) is $x^2 + y^2 + z^2 = 4$. On the xy-plane, $z = 0$.
The equation then becomes $x^2 + y^2 = 4$, which yields the parameterization $\boxed{\langle 2\cos t, -2\sin t, 0\rangle}$ for the clockwise direction.
2 rounds around the planet means that the domain of t is $\boxed{0 \le t \le 4\pi}$.
Since there is no restraint on where the curve starts, the answer can also be $\langle 2\sin t, 2\cos t, 0\rangle$ or $\langle -2\cos t, 2\sin t, 0\rangle$.