Center for Academic Resources in Engineering (CARE)
Peer Exam Review Session

Math 231 – Engineering Calculus

Midterm 1 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Sep. 21, 2:00pm-4:00pm Bella,

Session 2: Sep. 22, 1:00pm-3:00pm Grace, Sushrut

Can't make it to a session? Here's our schedule by course:

`https://care.grainger.illinois.edu/tutoring/schedule-by-subject`

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/844
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. Which of the following are equivalent to the following?

$$\frac{1}{4+i}$$

(a) $4-i$

(b) $i+4$

(c) $\frac{4}{17}-\frac{i}{17}$

(d) $\frac{4}{\sqrt{17}}-\frac{i}{\sqrt{17}}$

To start off, we need to multiply by the conjugate of

$$\frac{1}{4+i}$$

$$\frac{1}{4+i}*\frac{4-i}{4-i}=\frac{4-i}{16+1}$$

$$\frac{4-i}{16+1}=\boxed{\frac{4}{17}-\frac{i}{17}}$$

2. What is the simplified form of the following?

$$\left(\frac{2+i}{1+4i}\right)^2$$

(a) $\frac{3}{8i-15}$

(b) $\frac{24}{5}+\frac{7}{5}i$

(c) $-\frac{20}{46}-\frac{30}{46}i$

(d) $\frac{-13}{289}-\frac{84}{289}i$

To start solving this problem, we need to multiply by the conjugate.

$$\frac{2+i}{1+4i}*\frac{1-4i}{1-4i}=\frac{6-7i}{17}$$

But we still need to square this value!

$$\left(\frac{6-7i}{17}\right)^2=\frac{36-84i-49}{289}$$

$$\frac{36 - 84i - 49}{289} = \boxed{\frac{-13}{289} - \frac{84i}{289}}$$

3. Compute the Limit:

$$\lim_{x \to\ 0+} \frac{2}{x}$$

(a) $-\infty$

(b) 0

(c) does not exist

(d) ∞

As the function approaches 0 from the Right Hand Side, the limit approaches $\boxed{+\infty}$. For simple limit problems like this example, no work needs to be shown... Just your reasoning. :)

4. Compute the Limit:

$$\lim_{x \to \infty} \frac{2x + \sin(x)}{x}$$

(a) 2

(b) $\frac{1}{4}$

(c) ∞

(d) 0

$$\lim_{x \to \infty} \frac{2x + \sin(x)}{x} = \lim_{x \to \infty} 2 + \frac{\sin(x)}{x}$$

$$\lim_{x \to \infty} 2 + \frac{\sin(x)}{x} = 2 + \lim_{x \to \infty} \frac{\sin(x)}{x}$$

The rest of this problem can be handled with the squeeze theorem

$$\lim_{x \to \infty} \frac{\sin(x)}{x} = \lim_{x \to\ 0+} x \sin\left(\frac{1}{x}\right)$$

For $x > 0$

$$-1 < \sin\left(\frac{1}{x}\right) < 1$$

$$-x < x \sin\left(\frac{1}{x}\right) < x$$

$$\lim_{x \to\ 0+} x = 0$$

So the Squeeze Theorem tells us this limit equals zero.

BUT! We still have a value of 2 from earlier so this limit equals $\boxed{2}$

5. Compute the Limit:

$$\lim_{x\to\infty} \frac{\sin(5x)}{x}$$

(a) ∞

(b) 5

(c) does not exist

(d) 0

First, we can use the definition of a limit at infinity

$$\lim_{x\to\infty} \frac{\sin(5x)}{x} = \lim x \to 0+ \frac{\sin\left(\frac{5}{x}\right)}{\frac{1}{x}}$$

Now, we can use the squeeze theorem. Remember, sine fluctuates between -1 and 1

$$-1 < \sin\left(\frac{5}{x}\right) < 1$$

$$-x < x\sin\left(\frac{5}{x}\right) < x$$

So.... $\lim_{x\to 0+} x = \boxed{0}$

6. Compute the Limit:

$$\lim_{x\to\infty} \frac{9x^{6968} + 27x^{1789} + 88x^{900}}{45x^{6968} + 54x^{1789} + 264x^{900}}$$

(a) $\frac{1}{3}$

(b) $\frac{1}{2}$

(c) $\frac{1}{5}$

(d) ∞

First, we need to divide both the numerator and denominator by x^{6969}

$$\lim_{x\to\infty} \frac{9x^{6968} + 27x^{1789} + 88x^{900}}{45x^{6968} + 54x^{1789} + 264x^{900}} ==> \lim_{x\to\infty} \frac{9 + 27x^{-5180} + 88x^{-6069}}{45 + 54x^{-5180} + 264x^{-6069}}$$

$$\lim_{x\to\infty} \frac{9}{45} = \boxed{\frac{1}{5}}$$

7. Compute the Limit:

$$\lim_{x\to 3} \cos(3x)$$

(a) does not exist
(b) $\cos(3)$
(c) $\cos(9)$
(d) 1

Note that this is a continuous function. Knowing this, we can directly plug and chug

$$\lim_{x\to 3} \cos(3x) = \cos(3 * 3) = \boxed{\cos(9)}$$

8. If $f(y) = y\ln(y) - y$, compute $f'(1)$.

(a) 0
(b) ∞
(c) 1
(d) does not exist

$$\frac{d}{dy}(y\ln(y) - y) = \ln(y) + y * \frac{1}{y} - 1 = \ln(y)$$

Now, evaluate at y =1

$$\ln(1) = \boxed{0}$$

9. Compute $f'(x)$ where $f(x) = x\cos(x^4)$

(a) $-4x^4\sin(x^4)$
(b) $\sin(x^4) - 4x^4\cos(x^4)$
(c) $\cos(x^4) - 4x^4\sin(x^4)$
(d) $\cos(x^4)$

We will need to use the product and chain rules for this problem

$$\frac{d}{dx}(x\cos(x^4)) = \cos(x^4) * 1 + x(\frac{d}{dx}(\cos(x^4)))$$

$$\frac{d}{dx}(x\cos(x^4)) = \cos(x^4) + x(-4x^3\sin(x^4))$$

$$\frac{d}{dx}(x\cos(x^4)) = \boxed{\cos(x^4) - 4x^4\sin(x^4)}$$

10. If $f(x) = \Big((x^2+1)^3 + 4x\Big)^2$, them find $f'(1)$.

(a) 144

(b) 672

(c) 512

(d) 48

$$\frac{d}{dx}((x^2+1)^3+4x)^2 = 2((x^2+1)^3+4x)(3(x^2+1)^2(2x)+4)$$

$$f'(x) = 2((x^2+1)^3+4x)(6x(x^2+1)^2+4)$$

Now, substitute 1 in for x

$$f'(1) = 2((1^1+1)^3+4(1))(6(1)(1^2+1)^2+4)$$

$$f'(1) = 2(12)(28) = 672$$

$$f'(1) = \boxed{672}$$

11. If $3^x + 3^y = 3^{x+y}$ find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

(a) 0

(b) 3^{x-y}

(c) $\dfrac{3^{x+y} - 3^x}{3^y - 3^{x+y}}$

(d) $\dfrac{3^x - 3^y}{3^{x+y}}$

Differentiate both sides of $3^x + 3^y = 3^{x+y}$ w.r.t x to get $3^x + 3^y\dfrac{dy}{dx} = 3^{x+y} + 3^{x+y}\dfrac{dy}{dx}$.

Rearrange to get $\boxed{\dfrac{3^{x+y} - 3^x}{3^y - 3^{x+y}}}$.

12. A tank of water in the shape of a cone is being filled with water at a rate of $12\text{m}^3/\text{sec}$. The base radius of the tank is 26 meters and the height of the tank is 8 meters. At what rate is the depth of the water in the tank changing when the radius of the top of the water is 10 meters?

(a) $\dfrac{25\pi}{3}$

(b) $\dfrac{\pi}{3}$

(c) $\dfrac{25}{\pi}$

(d) $\dfrac{3}{25\pi}$

The volume of a cone is

$$V = \frac{1}{3}\pi r^2 h$$

Now,

$$\frac{r}{h} = \frac{26}{8} \implies r = \frac{13}{4}h \implies V = \frac{1}{3} * \frac{169}{16}\pi h^3$$

Differentiate to get

$$V' = \frac{169}{16}\pi h^2 h'$$

Since $r = 10$ we know that

$$h = \tfrac{4}{13}r = \tfrac{4}{13}(10) = \tfrac{40}{13}$$

Finally plug and chug to get,

$$12 = \frac{169}{16}\pi\left(\frac{40}{13}\right)^2 h' = 100\pi h' \Rightarrow \boxed{h' = \frac{3}{25\pi}}$$

13. The Maclaurin series expansion of $e^{\sin x}$ is:

(a) $1 + x - \frac{x^2}{2} + \frac{x^4}{12} - \cdots$

(b) $1 - x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots$

(c) $1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots$

(d) $1 + x + \frac{x^2}{2} - \frac{x^4}{12} + \cdots$

Let $f(x) = e^{\sin x}$.

$$\begin{aligned}
f'(x) &= e^{\sin x}\cos x = f(x)\cos x \\
f''(x) &= f'(x)\cos x - f(x)\sin x \\
f'''(x) &= f''(x)\cos x - f'(x)\sin x - (f(x)\cos x + f'(x)\sin x) \\
&= f''(x)\cos x - 2f'(x)\sin x - f'(x) \\
f^4(x) &= f'''(x)\cos x - f''(x)\sin x - 2(f'(x)\cos x + f''(x)\sin x) - f''(x) \\
&= f'''(x)\cos x - 3f''(x)\sin x - 2f'(x)\cos x - f''(x)
\end{aligned}$$

Calculating higher order derivatives at $x = 0$,

$$\begin{aligned}
f(0) &= 1 \\
f'(0) &= 1 \times 1 = 1 \\
f''(0) &= 1 - 0 = 1 \\
f'''(0) &= 1 - 0 - 1 = 0 \\
f^4(0) &= 0 - 0 - 2 - 1 = -3 \\
f^5(0) &= f^4(0)\cos 0 - 3f''(0)\cos 0 - 2f''(0)\cos 0 - f'''(0) \\
&= -3 - 3 - 2 - 0 = -8
\end{aligned}$$

Using Maclaurin's expansion for infinite series,

$$\begin{aligned}
f(x) &= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \\
&\quad \frac{x^4}{4!}f^4(0) + \frac{x^5}{5!}f^5(0) + \frac{x^6}{6!}f^6(0) + \ldots \\
e^{\sin x} &= 1 + x + \frac{x^2}{2!} \times 1 + 0 + \frac{x^4}{4!} \times (-3) + \frac{x^5}{5!} \times (-8) \ldots \\
&= \boxed{1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^5}{15} \ldots}
\end{aligned}$$