ILLINOIS

Center for Academic Resources in Engineering (CARE)
Peer Exam Review Session

Math 221 – Calculus I

Midterm 1 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Sep. 22nd, 5:00pm - 6:30pm -Zac, Sophia

Session 2: Sep. 23rd, 5:00pm - 6:30pm -Meredith, Erik

Can't make it to a session? Here's our schedule by course:

https://care.grainger.illinois.edu/tutoring/schedule-by-subject

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/1056
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. Let $f(x) = \frac{2}{\sqrt{x-1}}$. Use the definition of a derivative as a limit to prove that $f'(x) = \frac{-1}{(x-1)^{3/2}}$. Show each step in your calculation and be sure to use proper terminology in each step of your proof.

$$f'(x) = \lim_{h\to 0} \frac{\frac{2}{\sqrt{x+h-1}} - \frac{2}{\sqrt{x-1}}}{h} = \lim_{h\to 0} \frac{2\left(\frac{\sqrt{x-1}-\sqrt{x+h-1}}{\sqrt{x+h-1}\cdot\sqrt{x-1}}\right)}{h}$$

Next, multiply the numerator and denominator by the conjugate of the numerator $\sqrt{x-1} + \sqrt{x+h-1}$:

$$f'(x) = \lim_{h\to 0} \frac{2\left(\frac{-h}{\left(\sqrt{x+h-1}\cdot\sqrt{x-1}\right)\left(\sqrt{x-1}+\sqrt{x+h-1}\right)}\right)}{h} = \lim_{h\to 0} \frac{-2}{\sqrt{x+h-1}\cdot\sqrt{x-1}\cdot\left(\sqrt{x-1}+\sqrt{x+h-1}\right)}$$

$$f'(x) = \frac{-2}{\sqrt{x-1}\cdot\sqrt{x-1}\cdot 2\sqrt{x-1}} = \frac{-2}{2(x-1)^{3/2}} = \frac{-1}{(x-1)^{3/2}}$$

2. Compute the following limit:

$$\lim_{x\to\infty} \frac{91\sqrt[8]{x}+3}{5-7\sqrt[8]{x}}$$

$$\lim_{x\to\infty} \frac{\sqrt[8]{x}(91+\frac{3}{\sqrt[8]{x}})}{\sqrt[8]{x}\left(\frac{5}{\sqrt[8]{x}}-7\right)}$$

$$\lim_{x\to\infty} \frac{91+\frac{3}{\sqrt[8]{x}}}{\frac{5}{\sqrt[8]{x}}-7}$$

$$\frac{91+0}{0-7} = \boxed{-13}$$

3. Write an equation for each horizontal asymptote on the graph of the following function. Use limits to justify your answer. We will learn L'Hopital's Rule and other shortcuts for obtaining limits later. For now, you are not allowed to use these approaches.

$$\frac{56e^{-5x}-30}{7e^{-5x}+10}$$

$$\lim_{x\to\infty} \frac{56e^{-5x}-30}{7e^{-5x}+10} = \frac{56*0-30}{7*0+10} = -3$$

$$\lim_{x\to-\infty}\frac{56e^{-5x}-30}{7e^{-5x}+10}=\frac{56}{7}=8$$

So the equations of the asymptotes are $\boxed{\text{y = -3}}$ and $\boxed{\text{y = 8}}$. A helpful hint for this problem is to think about the graph of $y=e^{-5x}$.

4. Compute the following limits:

(a)

$$\lim_{x\to 0}\frac{19x-5sin(x)}{2x}$$

(b)

$$\lim_{x\to 0}\frac{e^{6x}-1}{e^{3x}-1}$$

(a)

$$lim_{x\to 0}(\frac{19x}{2x}-\frac{5\sin(x)}{2x})$$

$$\lim_{x\to 0}(\frac{19}{2}-\frac{5}{2}\frac{\sin(x)}{x})=\frac{19}{2}-\frac{5}{2}=\boxed{7}$$

* Remember that $\lim_{x\to 0}\frac{\sin(x)}{x}=1$

(b)

$$\lim_{x\to 0}\frac{(e^{3x})^2-1}{e^{3x}-1}=\lim_{x\to 0}\frac{(e^{3x}-1)(e^{3x}+1)}{e^{3x}-1}$$

$$\lim_{x\to 0}e^{3x}+1=e^{3*0}+1=1+1=\boxed{2}$$

5. Determine whether the following statements are always true.

(a) A function which is continuous at point (a) must also be differentiable at (a).

(b) The limit $\lim_{x \to a} f(x)$ depends only on the values of the function near a, not $f(a)$ itself.

(c) The function $y = \frac{9x-63}{x^2+6x-91}$ has a vertical asymptote at $x = 7$.

(a) This statement is **false**. A function which is differentiable at (a) must be continuous at (a), but not the other way around. For example, $F(x) = |x - 2|$ is continuous but not differentiable at 2.

(b) This statement is **true**. The value of $\lim_{x \to a} f(x)$ depends on the behavior of $f(x)$ as x approaches a from both sides. The function doesn't need to be defined at a for the limit to exist; only the behavior as $x \to a$ matters.

(c) This statement is **false**.

$lim_{x \to 7} \frac{9x-63}{x^2+6x-91} = \lim_{x \to 7} \frac{9(x-7)}{(x-7)(x+13)}$

$\lim_{x \to 7} \frac{9}{x+13} = \frac{9}{20}$

Clearly this does not go to +/- ∞ so there is no asymptote.

6. A car starts from rest and travels along a straight highway for 30 minutes. At some point during the trip, the car reaches a speed of 60 miles per hour. The car's speed is a continuous function of time. Prove that there is a moment during the trip when the car's speed was exactly 30 miles per hour. (Hint: Use Intermediate Value Theorem)

Let $v(t)$ represent the car's speed as a continuous function of time t, where:

$$v(0) = 0 \quad \text{and} \quad v(T) = 60 \quad \text{for} \quad T = 30 \text{ minutes.}$$

We want to show that there exists $c \in (0, T)$ such that $v(c) = 30$.

Since $v(t)$ is continuous and $v(0) = 0$ and $v(T) = 60$, by the Intermediate Value Theorem (IVT), for any value between 0 and 60, there must exist $c \in (0, T)$ such that:

$$v(c) = 30.$$

Thus, at some point during the trip, the car's speed was exactly 30 miles per hour.

7. Evaluate the following limits and write your answers in simplified form.

(a)

$$\lim_{x \to \sqrt{2}} \frac{120 \arcsin\left(\frac{x}{2}\right)}{x^2 + 4}$$

(b)

$$\lim_{x\to\infty} \frac{13 + 5\sin(9e^{3x} + 6)}{x^{10}}$$

(a) First, we can directly plug in the value of the limit:

$$\frac{120 \arcsin\left(\frac{\sqrt{2}}{2}\right)}{(\sqrt{2})^2 + 4}$$

$$\frac{120(\frac{\pi}{4})}{2 + 4}$$

$$\frac{120(\frac{\pi}{4})}{6}$$

$$20(\frac{\pi}{4}) = \boxed{5\pi}$$

(b) Within this specific function, the range of sine fluctuates between -1 and 1:
$-1 < \sin(9e^{3x} + 6) < 1$: Now multiply by 5

$-5 < 5\sin(9e^{3x} + 6) < 5$: Add 13 to all sides

$8 < 13 + 5\sin(9e^{3x} + 6) < 18$: Divide all sides by x^{10}

$$\frac{8}{x^{10}} < \frac{13 + 5\sin(9e^{3x} + 6)}{x^{10}} < \frac{18}{x^{10}}$$

$$\lim_{x\to\infty} \frac{8}{x^{10}} = 0 \text{ and } \lim_{x\to\infty} \frac{18}{x^{10}} = 0$$

So, by the squeeze theorem, it must be true that:

$$\lim_{x\to\infty} \frac{13 + 5sin(9e^{3x} + 6)}{x^{10}} = \boxed{0}$$

8. Use implicit differentiation to find $\frac{dy}{dx}$ for the equation $xy + y^4 = x^3y$.
Remember to treat y as a function of x (i.e., use implicit differentiation).

$$y + x\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 3x^2y + x^3\frac{dy}{dx}$$

$$x\frac{dy}{dx} + 4y^3\frac{dy}{dx} - x^3\frac{dy}{dx} = 3x^2y - y$$

$$\frac{dy}{dx}\left(x+4y^3-x^3\right)=3x^2y-y$$

$$\frac{dy}{dx}=\frac{3x^2y-y}{x+4y^3-x^3}$$

9. Compute the following derivatives:
a.)

$$y=\frac{x^2+1}{x-1}$$

This is a derivative that can be solved using the quotient rule!

$$\frac{dy}{dx}=\frac{\frac{d}{dx}(x^2+1)(x-1)-\frac{d}{dx}(x-1)(x^2+1)}{(x-1)^2}$$

$$\frac{dy}{dx}=\frac{(2x)(x-1)-1(x^2+1)}{(x-1)^2}$$

$$\boxed{\frac{dy}{dx}=\frac{x^2-2x-1}{(x-1)^2}}$$

b.)

$$f(x)=sec^{-1}(x^4)$$

This is a derivative that can be solved by using chain rule!

(a) Let's first start by finding the derivatives of the inside and outside functions :

$$\frac{d}{dx}(x^4)=4x^3$$

Recall the derivatives of inverse trig functions...

$$\frac{d}{dx}(sec^{-1}(x))=\frac{x'}{|x|\sqrt{x^2-1}}$$

(b) Putting it all together :

$$f'(x)=\frac{1}{x^4\sqrt{x^8-1}}\cdot 4x^3$$

$$f'(x)=\frac{4x^3}{x^4\sqrt{x^8-1}}$$

$$\boxed{f'(x) = \frac{4}{x\sqrt{x^8 - 1}}}$$

c.)

$$f(x) = \frac{3x}{5 - sec(x)}$$

This is derivative that can be solved by applying the quotient rule again!

$$f'(x) = \frac{\frac{d}{dx}(3x)(5 - sec(x)) - \frac{d}{dx}(sec(x))(3x)}{(5 - sec(x))^2}$$

$$\boxed{f'(x) = \frac{3(5 - sec(x)) - (-sec(x)tan(x))3x}{(5 - sec(x))^2}}$$

d.)

$$y = \frac{cos(4x) - 1}{e^x x^3}$$

This is a derivative that can be solved by using a combination of quotient and chain rule!

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(cos(4x) - 1)(e^x x^3) - \frac{d}{dx}(e^x x^3)(cos(4x) - 1)}{(e^x x^3)^2}$$

(a)

$$\frac{d}{dx}(cos(4x) - 1) = -4sin(4x)$$

$$\frac{d}{dx}(e^x x^3) = e^x x^3 + 3x^2 e^x$$

(b) Putting it all together:

$$\frac{dy}{dx} = \frac{-4sin(4x)(e^x x^3) - (e^x x^3 + 3x^2 e^x)(cos(4x) - 1)}{(e^x x^3)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{-4sin(4x)(e^x x^3) - (e^x x^3 + 3x^2 e^x)(cos(4x) - 1)}{e^{2x} x^6}}$$