Center for Academic Resources in Engineering (CARE)
Peer Exam Review Session

MATH 257 – Linear Algebra with Computational Applications

Midterm 1 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Sept. 18, 6:30-8:30pm Alice, JD, Ryan

Session 2: Sept. 19, 7-9pm Danielle, Cain, Rishi

Can't make it to a session? Here's our schedule by course:

https://care.grainger.illinois.edu/tutoring/schedule-by-subject

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/955
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. Consider the system of linear equations:

$$\begin{cases} x_1 + 4x_2 + 2x_3 & = 2 \\ x_1 + 4x_2 + \quad +2x_4 & = 2 \end{cases}$$

a) Create an augmented matrix for this system and put it into reduced row echelon form.

Create the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 4 & 2 & 0 & 2 \\ 1 & 4 & 0 & 2 & 2 \end{array}\right]$$

Find the row echelon form of the matrix

$$\left[\begin{array}{cccc|c} 1 & 4 & 2 & 0 & 2 \\ 1 & 4 & 0 & 2 & 2 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 4 & 2 & 0 & 2 \\ 0 & 0 & -2 & 2 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 0 & 2 & 2 \\ 0 & 0 & -2 & 2 & 0 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{2} R_2} \left[\begin{array}{cccc|c} 1 & 4 & 0 & 2 & 2 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \to -R_2}$$

$$\boxed{\left[\begin{array}{cccc|c} 1 & 4 & 0 & 2 & 2 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]}$$

b) Which variables are basic? Which variables are free?

$$\begin{cases} x_1 = & 2 - 4x_2 - 2x_4 \\ x_2 = & x_2 \\ x_3 = & x_4 \\ x_4 = & x_4 \end{cases}$$

Free variables can be any value and do not depend on other variables. Basic variables, or pivot variables can be written in terms of the free variables using the equations given by row echelon form.

$\boxed{\text{The basic variables are } x_1 \text{ and } x_3}$

$\boxed{\text{The free variables are } x_2 \text{ and } x_4}$

c) State the general solution in **parametric** form.

Write the general solution in terms of the free variables only.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

2. Let $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ be defined as follows:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \ \boldsymbol{B} = \begin{bmatrix} 4 & 1 \\ 6 & 0 \end{bmatrix} \text{ and } \boldsymbol{C} = \begin{bmatrix} 1 & 0 \\ 0 & 8 \end{bmatrix}$$

Compute the following:

a) $\boldsymbol{A} + \boldsymbol{C}$

Add the elements:

$$\boxed{\begin{bmatrix} 2 & 2 \\ 0 & 11 \end{bmatrix}}$$

b) $\boldsymbol{2B} + \boldsymbol{A}$

First compute $\boldsymbol{2B}$ then add:

$$\begin{bmatrix} 8 & 2 \\ 12 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \boxed{\begin{bmatrix} 9 & 4 \\ 12 & 3 \end{bmatrix}}$$

c) $\boldsymbol{4C^T} - \boldsymbol{B}$

C is symmetric, so $\boldsymbol{C^T} = \boldsymbol{C}$. Then compute $\boldsymbol{4C^T}$, finally subtract.

$$\begin{bmatrix} 4 & 0 \\ 0 & 32 \end{bmatrix} - \begin{bmatrix} 4 & 1 \\ 6 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & -1 \\ -6 & 32 \end{bmatrix}}$$

d) $\boldsymbol{C} + \frac{\boldsymbol{1}}{\boldsymbol{2}}\boldsymbol{B^T} - \boldsymbol{A}$

$$\frac{\boldsymbol{1}}{\boldsymbol{2}}\boldsymbol{B^T} = 1/2 \begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1/2 & 0 \end{bmatrix}$$

Compute the rest of the problem:

$$\begin{bmatrix} 1 & 0 \\ 0 & 8 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 1/2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 1 \\ 1/2 & 5 \end{bmatrix}}$$

3. Let $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ be defined as follows:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \ \boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & 0 & 1 \end{bmatrix} \text{ and } \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 8 & 7 \end{bmatrix}$$

a) Compute the following or state it is not defined:

i) $\boldsymbol{BC}$

The dimensions of $\boldsymbol{B}$ are 3×3 but the dimensions of $\boldsymbol{C}$ are 2×3. The columns of $\boldsymbol{B}$ and the rows of $\boldsymbol{C}$ do not match, therefore we cannot compute their product

$$\boxed{\text{It is not defined.}}$$

ii) $\boldsymbol{ACB}$

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 8 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & 0 & 1 \end{bmatrix}$$

Compute AC first:

$$\begin{bmatrix} 1*1+2*0 & 1*0+2*8 & 1*1+2*7 \\ 0*1+3*0 & 0*0+3*8 & 0*1+3*7 \end{bmatrix} = \begin{bmatrix} 1 & 16 & 15 \\ 0 & 24 & 21 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 16 & 15 \\ 0 & 24 & 21 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1*1+16*0+15*6 & 1*0+16*1+15*0 & 1*0+16*0+15*1 \\ 0*1+24*0+21*6 & 0*0+24*1+21*0 & 0*0+24*0+21*1 \end{bmatrix} =$$

$$\boxed{\begin{bmatrix} 91 & 16 & 15 \\ 126 & 24 & 21 \end{bmatrix}}$$

b) The matrix $\boldsymbol{B}$ is an elementary matrix. What row operation would $\boldsymbol{B}$ do if it was multiplied on the left to a matrix $\boldsymbol{M}$ of an appropriate size?

The matrix $\boldsymbol{B}$ has a 6 in the third row, meaning it will scale the third row by a factor of 6. Thus, the matrix $\boldsymbol{BM}$ is the matrix you would get if you did the elementary row operation:

$$\boxed{R_3 \to R_3 + 6R_1}$$

4. Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be three nonzero vectors in $\mathbb{R}^3$ and let $\mathbf{c}$ be in $\text{span}\{\mathbf{a}, \mathbf{b}\}$.

(a) Describe span$\{\mathbf{a}, \mathbf{b}\}$

The span of the two vectors would be the plane containing both vectors **a** and **b** that also passes through the origin.

(b) Suppose these three vectors are arranged in a matrix $\boldsymbol{A}$ column-wise. How many pivots can $\boldsymbol{A}$ have at most?

Since **c** is in span$\{\mathbf{a}, \mathbf{b}\}$, there can only be two linearly independent vectors. Therefore, there can at most be two pivots because one of the columns can be written as a linear combination of the others.

5. True or False: A square matrix $\boldsymbol{A}$ is invertible if and only if $\det \boldsymbol{A} = 0$.

   A matrix is invertible if and only if it is square and has a nonzero determinant. If the determinant is zero, the matrix is not invertible. This is because if any of the columns/rows of the matrix are zero or if one column/row can be written as a linear combination of the other columns/rows, the determinant will be zero.

   $$\boxed{\text{False}}$$

6. Let $\boldsymbol{A}$ be an $m \times n$ matrix. What are the dimensions of $\boldsymbol{A}^T \boldsymbol{A}$?

   Firstly, the dimension of $\boldsymbol{A}^T$ would be $n \times m$
   Multiplying an $n \times m$ matrix ($\boldsymbol{A}^T$) by an $m \times n$ matrix ($\boldsymbol{A}$) yields a square matrix of size:

   $$\boxed{n \times n}$$

7. Consider a linear system $\boldsymbol{Ax} = \mathbf{b}$. Given that the row reduced echelon form of $\boldsymbol{A}$ is

   $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   Which of the following are true?

I) The system has exactly one solution

II) The system has no solutions

III) There are three pivots associated with $\boldsymbol{A}$

IV) $\boldsymbol{A}$ is invertible

   Since each column has one nonzero entry in row reduced echelon form, $\boldsymbol{A}$ has three pivots and no free variables. This also means that it is invertible, since having three pivots means each row is linearly independent. Therefore, there is also one unique solution. The correct options are $\boxed{\text{I, III, and IV}}$.

8. Which of the following matrices is the inverse to $\boldsymbol{A}$?

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) $\begin{bmatrix} 0 & 1 & -2 & -2 \\ -1 & 1 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & -1 & 2 & 0 \\ 1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -1 & -2 & 0 \\ -1 & 1 & -2 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

First augment the matrix with the identity

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Subtract the first row from the second row $R_2 \to R_2 - R_1$

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Add the second row to the first row $R_1 \to R_1 + R_2$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Subtract twice the third row from the second row $R_2 \to R_2 - 2R_3$
Subtract twice the third row from the first row $R_1 \to R_1 - 2R_3$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Subtract twice the fourth row from the first row $R_1 \to R_1 - 2R_4$
Add the fourth row to the second row $R_2 \to R_2 + R_4$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & -2 & -2 \\ 0 & 1 & 0 & 0 & -1 & 1 & -2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Since we have the identity matrix on the left now we have found that the inverse of $\boldsymbol{A}$ is $\boxed{(a)}$