Q3 A positron of mass m and charge $+e$ interacts with a magnetic field $\mathbf{B}$ through the Hamiltonian

$$\hat{H} = -\frac{e}{m}\hat{\mathbf{S}} \cdot \mathbf{B}(t),$$

where $\hat{\mathbf{S}} = (\hbar/2)(\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$, and the $\hat{\sigma}_i$ are the Pauli matrices

$$\hat{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \hat{\sigma}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \hat{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

a) Given a Schrödinger-picture state $|\psi(t)\rangle$ that evolves under $\hat{H}$, show that the vector of expectation values

$$\mathbf{S}(t) = (\langle\psi|\hat{S}_x|\psi\rangle, \langle\psi|\hat{S}_y|\psi\rangle, \langle\psi|\hat{S}_z|\psi\rangle),$$

obeys

$$\frac{d\mathbf{S}(t)}{dt} = \alpha \mathbf{S}(t) \times \mathbf{B}(t), \quad (\star)$$

where α is a constant that you should find.

b) In the Heisenberg picture the time-independent $\hat{\mathbf{S}}$ is replaced by a time-dependent operator $\hat{\mathbf{S}}(t)$. Show that $\hat{\mathbf{S}}(t)$ obeys an operator equation of similar form to $(\star)$, and from it show that the vector of expectation values $\mathbf{S}(t)$ still obeys $(\star)$.

c) Suppose the Schrödinger-picture state $|\psi(t)\rangle$ at $t = 0$ is described in the eigenstate basis of $\hat{\sigma}_z$ by

$$\langle +1|\psi\rangle = \cos(\theta/2), \quad \langle -1|\psi\rangle = e^{i\phi}\sin(\theta/2).$$

Compute the vector $\mathbf{S}(t = 0)$ and explain the relation of the parameters θ, ϕ to the direction of the spin.

d) Let $\mathbf{B}$ vary linearly with time so $\mathbf{B}(t) = (0, 0, Ft)$ where F is constant. Given the initial state from part (c), find $|\psi(t)\rangle$ and hence $\mathbf{S}(t)$. Does it satisfy $(\star)$ from part (a)?