Center for Academic Resources in Engineering (CARE) Peer Exam Review Session

Math 285 – Intro Differential Equations

Midterm 3 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Wed., Apr. 24th 6:00-7:30pm 2039 CIF Tutors: Hayden and Suleymaan

Session 2: Thurs., Apr. 25th 5:00-6:30pm 1035 CIF Tutors: Charlie and Eric

Can't make it to a session? Here's our schedule by course:

`https://care.grainger.illinois.edu/tutoring/schedule-by-subject`

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/846
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. Find the eigenvalues and corresponding eigenvectors for the following matrix A:

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

The eigenvalues can be calculated with the following equation: $\det(A - I) = 0$

$$\begin{vmatrix} (0-\lambda) & 1 \\ -2 & (-3-\lambda) \end{vmatrix} = 0 \rightarrow (-\lambda)(-3-\lambda) - (-2) = 0$$

$$\lambda^2 + 3\lambda + 2 = 0 \rightarrow \quad \lambda = -1, -2$$

To find eigenvectors, the following equation must be satisfied: $(A - \lambda) * V = 0$

For $\lambda = -1$:

$$\begin{bmatrix} (0-(-1)) & 1 \\ -2 & (-3-(-1)) \end{bmatrix} * \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Multiplying the matrix by the vector results in the equation $a_1 = -a2$. If a_2 is assumed to be 1, then the eigenvalue $\lambda = -1$ has eigenvector:

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

For $\lambda = -2$:

$$\begin{bmatrix} (0-(-2)) & 1 \\ -2 & (-3-(-2)) \end{bmatrix} * \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Multiplying the matrix by the vector results in the equation $a_2 = -2a_1$. If a_1 is assumed to be 1, then the eigenvalue $\lambda = -2$ has eigenvector:

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

2. Find the matrix exponential using Putzer's Method for the following matrix A:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 2 & 0 & 0 \end{bmatrix}$$

Putzer's Method requires the eigenvalues of the matrix. These can be found with the equation $det(A - \lambda I) = 0$.

$$\begin{vmatrix} (1-\lambda) & 0 & 0 \\ 0 & (-1-\lambda) & 2 \\ 2 & 0 & (-\lambda) \end{vmatrix} = 0$$

$$(1-\lambda) \cdot [(-1-\lambda) \cdot (-\lambda) - (2 \cdot 0)] = 0$$

$$(1-\lambda) \cdot (-1-\lambda) \cdot (-\lambda) = 0$$

The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = -1$.

Next, set up the corresponding B_0, B_1, and B_2 matrices:

$$B_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B_1 = A - \lambda_1 I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 2 & 0 & 0 \end{bmatrix}$$

$$B_2 = (A - \lambda_2 I)(A - \lambda_1 I) = \begin{bmatrix} (1-1) & 0 & 0 \\ 0 & (-1-1) & 2 \\ 2 & 0 & (0-1) \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 2 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, we must solve a series of first order ODEs to find r_1, r_2, and r_3 so Putzer's definition of the matrix exponential can be used.

(a) r_1 (Separable equation)

$$\frac{dr_1}{dt} = \lambda_1 r_1 \quad r_1(0) = 1$$

$$\frac{dr_1}{dt} = 0r_1 \to r_1 = c \to \boxed{r_1 = 1}$$

(b) r_2 (Integrating factor)

$$\frac{dr_2}{dt} = \lambda_2 r_2 + r_1 \quad r_2(0) = 0$$

$$\frac{dr_2}{dt} = 1r_2 + 1 \to \frac{dr_2}{dt} - r_2 = 1$$

$$r_2 = -1 + ce^t \to r_2(0) = 0 \to \boxed{r_2 = e^t - 1}$$

(c) r_3 (Integrating factor)

$$\frac{dr_3}{dt} = \lambda_3 r_3 + r_2 \quad r_3(0) = 0$$

$$\frac{dr_3}{dt} = -r_3 + e^t - 1 \to \frac{dr_3}{dt} + r_3 = e^t - 1$$

$$r_3 = \frac{1}{2}e^t - 1 + ce^{-t} \to r_3(0) = 0 \to \boxed{r_3 = \frac{1}{2}e^t + \frac{1}{2}e^{-t} - 1}$$

Now use Putzer's definition of the matrix exponential:

$$e^{At} = B_0 \cdot r_1 + B_1 \cdot r_2 + B_2 \cdot r_3$$

$$= 1 \cdot B_0 + (e^t - 1) \cdot B_1 + (\frac{1}{2}e^t + \frac{1}{2}e^{-t} - 1) \cdot B_2$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} (e^t - 1) & 0 & 0 \\ 0 & (1 - e^t) & (2e^t - 2) \\ (2e^t - 2) & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ (2e^t + 2e^{-t} - 4) & (e^t + e^{-t} - 2) & (-2e^t - 2e^{-t} + 4) \\ 0 & 0 & 0 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} (e^t) & 0 & 0 \\ (2e^t + 2e^{-t} - 4) & (e^{-t}) & (-2e^{-t} + 2) \\ (2e^t - 2) & 0 & 1 \end{bmatrix}$$

3. Solve the ODE with given boundary value conditions. How many solutions does it have?

$$y'' + y = 0 \qquad y(-\pi) = 0, \qquad y(\pi) = 2$$

The characteristic equation $r^2 + 1 = 0$ has roots $r = \pm i$, so

$$y_c = C_1 sin(t) + C_2 cos(t)$$

Substituting for the first condition,

$$y(-\pi) = C_1 sin(-\pi) + C_2 cos(-\pi)$$

$$0 = C_1(0) + C_2(-1)$$

$$C_2 = 0$$

Substituting for the second condition,

$$y(\pi) = C_1 sin(\pi) + C_2 cos(\pi)$$

$$2 = C_1(0) + C_2(-1)$$

$$C_2 = -2$$

Since C_2 can't equal both 0 and -2, $\boxed{\text{there are no solutions.}}$

4. Find the matrix exponential using diagonalization for the following matrix A:

$$\begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}$$

Diagonalization requires the eigenvalues and eigenvectors of the matrix. These can be found with the equation $det(A - \lambda I) = 0$.

$$\begin{vmatrix} (5-\lambda) & 4 \\ 2 & (3-\lambda) \end{vmatrix} = 0$$

$$(5-\lambda)\cdot(3-\lambda) - 8 = 0$$

$$\lambda^2 - 8 + 7 = 0 \Rightarrow (\lambda - 7)(\lambda - 1) = 0$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 7$.

Next, calculate the corresponding eigenvectors using the eigenvector definition $(A - \lambda I)\nu = 0$

For $\lambda_1 = 1$:

$$\begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

By inspection, $A + B = 0$, so:

$$\lambda_1 = 1, \quad \nu_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

For $\lambda_1 = 7$:

$$\begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

By inspection, $2A - 4B = 0$, so:

$$\lambda_2 = 7, \quad \nu_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Now, to perform diagonalization, create the matrices U and D:

$$U = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} e^t & 0 \\ 0 & e^{7t} \end{bmatrix}$$

Finally:

$$e^{At} = UDU^{-1}$$

$$e^{At} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{7t} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{-2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} \frac{e^t + 2e^{7t}}{3} & \frac{-2e^t + 2e^{7t}}{3} \\ \frac{-e^t + e^{7t}}{3} & \frac{2e^t + e^{7t}}{3} \end{bmatrix}$$

5. Match all the sets of boundary conditions with the solution type they produce when imposed on the following homogeneous ODE:

$$y'' - 2y' = 0$$

A) $y'(0) = 1$ and $y'(1) = 0$ (I) Unique Solution
B) $y'(1) = 2$ and $y''(1) = 4$ (II) Infinitely Many Solutions
C) $y(0) = 5$ and $y'(0) = 2$ (III) Trivial Solution
D) $y(0) = 0$ and $y'(0) = 0$ (IV) No Solution

The homogeneous ODE has the following characteristic equation and roots:

$$r^2 - 2r = r(r-2) = 0$$

Therefore,

$$y(x) = c_1 + c_2 * e^{2x}$$

Using the boundary conditions from A):

$$y'(x) = 2c_2e^{2x}$$
$$y'(0) = 1 \to 1 = 2c_2e^0 \to c_2 = 1/2$$
$$y'(1) = 0 \to 0 = 2c_2e^2 \to c_2 = 0$$

c_2 can't have two different values. Therefore, A) leads to (IV) No Solution since both boundary conditions can't be satisfied.

Using boundary conditions from B):

$$y''(x) = 4c_2e^{2x}$$
$$y'(1) = 2 \to 2 = 2c_2e^2 \to c_2 = e^{-2}$$
$$y''(1) = 4 \to 4 = 4c_2e^2 \to c_2 = e^{-2}$$

Both conditions specifiy that $c_2 = e^{-2}$, so c_1 can be any value and still satisfy both boundary conditions. Therefore, the B) boundary conditions lead to (II) Infinitely Many Solutions.

Using boundary conditions from C):

$$y'(0) = 2 \xrightarrow{2} = 2c_2e^0 \to c_2 = 1$$
$$y(0) = 5 \xrightarrow{5} = c1 + c^2e^0c_1 = 5 - c_2 = 5 - 1 = 4$$

Since unique values can be found for both c_1 and c_2 for the boundary conditions, the C) conditions lead to (I) Unique Solution.

Using boundary conditions from D):

$$y'(0) = 0 \to 0 = 2c_2e^0 \to c_2 = 0$$
$$y(0) = 0 \to 0 = c_1 + c_2c^0 \to 0 = c1 + 0 \to c_1 = 0$$

Since the boundary conditions require that both c_1 and c_2 are 0, the D) boundary conditions lead to (III) Trivial Solution.

6. Compute all the eigenvalues and corresponding eigenfunctions for the boundary value problem

$$y'' - \lambda y = 0 \qquad y'(-2) = 0, y(0) = 0$$

If a certain range of the real numbers does not include any eigenvalues, show why there are none in that range

The BVP has eigenvalues

$$\lambda_n = -\frac{(n-\frac{1}{2})^2\pi^2}{4}$$

with the corresponding eigenfunctions

$$y_n(x) = \sin\left(\frac{(n-\frac{1}{2})\pi x}{2}\right), \text{ (n = 1,2...)}$$

a. There are no positive eigenvalues. Write $\lambda = \mu^2$ (with $\mu > 0$), hence we are solving the diff eq

$$y'' - \mu y = 0$$

The corresponding characteristic equation $r^2 - \mu^2 = 0$ has roots $r = \pm\mu$ hence the general solution is

$$y(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}$$

Then $y(0) = C_1 + C_2 = 0$, hence $C_2 = -C_1$ and $y(x) = C_1(e^{\mu x} - e^{-\mu x})$
Further, $y'(x) = C_1\mu(e^{\mu x} + e^{-\mu x})$ so $y'(-2) = C_1\mu(e^{-2\mu} + e^{2\mu}) = 0$
This leads to $C_1 = 0$, hence $y(x) = 0$

b. Zero is not an eigenvalue either. For $\lambda = 0$ we are solving the DE $y'' = 0$
$y(x) = C_1 x + C_2$. We have $y(0) = C_1 = 0$ and $y'(-2) = C_2 = 0$, hence $y = 0$

c. Finally we look for negative eigenvalues λ = -μ^2, with $\mu > 0$. We are solving the diff eq

$$y'' + \mu^2 y = 0$$

The general solution has the form

$$y(x) = C_1\cos(\mu x) + C_2\sin(\mu x)$$

From $y(0) = y(0) = C_1$ we conclude that $y(x) = C_2\sin(\mu x)$ and $y'(x) = C_2\mu\cos(\mu x)$
We have $y'(-2) = C_2\mu\cos(2\mu) = 0$, hence y can be non-zero if and only if $\cos(2\mu) = 0$

This last equality occurs if and only if

$$2\mu = \pi\big(n - \frac{1}{2}\big)$$

for some positive integer n. Thus, we have eigenvalues

$$\boxed{\lambda_n = -\mu_n^2 = -\frac{(n-\frac{1}{2})^2\pi^2}{4}}$$

with the corresponding eigenfunctions

$$\boxed{y_n(x) = \sin(\mu_n x) = \sin\left(\frac{(n-\frac{1}{2})\pi x}{2}\right)}$$

7. Functions f, g, h and k are 6-periodic. Their values on $[-3,3)$ are given below. For which of these functions does the Fourier series converge at $x=0$ to the value 1?

$$f(x)=\begin{cases}2+x & -3\le x<0\\ 1 & x=0\\ -2 & 0<x<3\end{cases}\qquad g(x)=\begin{cases}1+x & -3\le x<0\\ 4 & x=0\\ 2-x^2 & 0<x<3\end{cases}$$

$$h(x)=\begin{cases}x^2-1 & -3\le x<0\\ -1 & x=0\\ 3 & 0\le x<3\end{cases}\qquad k(x)=\begin{cases}3+x & -3\le x<1\\ 1 & x=1\\ x-8 & 1<x<3\end{cases}$$

A) f

B) h

C) None

D) g and k

E) f and h

The Fourier series of a function $\phi(x)$ converges at $x=0$ to value $\phi(0)$ if ϕ is continuous at $x=0$, and converges to value $\frac{\phi(0-)+\phi(0+)}{2}$ if ϕ jumps at $x=0$. Here is the summary of relevant information

f: $f(0-)=2$, $f(0+)=-2$, jump at $x=0$, so Fourier series at 0 has value 0

g: $g(0-)=1$, $g(0+)=2$, jump at $x=0$, so Fourier series at 0 has value $\frac{3}{2}$

h: $h(0-)=-1$, $h(0+)=3$, jump at $x=0$, so Fourier series at 0 has value 1

k: $k(0)=3$ since k is continuous at $x=0$, so Fourier series at 0 has value 3

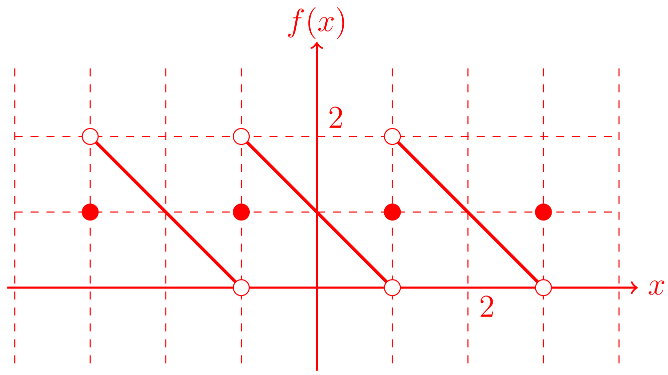
The answer is **(B)**.

8. Consider the function $f(x) = 1 - x$ defined on the interval $x \in [-1, 1)$

   (a) Sketch the 2-periodic Classical extension of f(x) on the interval $x \in [-3, 3]$

   (b) Compute the 2-periodic Classical Fourier series representation of f(x)

(a) The sketch is shown below.



(b) Here we have $L = 1$ so the Fourier expansion is

$$f(x) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi x}{1}\right) + B_n \sin\left(\frac{n\pi x}{1}\right) \right)$$

$$A_0 = \frac{1}{2L} \int_0^{2L} f(x)dx = \frac{1}{2} \int_{-1}^{1} (1-x)dx = \frac{1}{2}\left[x - \frac{x^2}{2}\right]_{-1}^{1} = 1$$

$$A_n = \frac{1}{L} \int_0^{2L} f(x) \cos\left(\frac{n\pi x}{L}\right)dx = \int_{-1}^{1} (1-x)\cos(n\pi x)dx$$

However, since $(1-x)\cos(n\pi x)$ is an odd function and the integration interval is symmetric:

$$A_n = 0$$

$$B_n = \frac{1}{L} \int_0^{2L} f(x) \sin\left(\frac{n\pi x}{L}\right)dx = \int_{-1}^{1} (1-x)\sin(n\pi x)dx = \int_{-1}^{1} \sin(n\pi x)dx - \int_{-1}^{1} x\sin(n\pi x)dx$$

$\int_{-1}^{1} \sin(n\pi x)dx = 0$ since integrating an odd function over a symmetric interval. The second term can be solved with Integration by Parts:

$$-\int_{-1}^{1} x\sin(n\pi x)dx = \left[\frac{x}{n\pi}\cos(n\pi x)\right]_{-1}^{1} - \frac{1}{n\pi}\int_{-1}^{1} \cos(n\pi x)dx$$

$$\frac{2\cos(n\pi)}{n\pi} - \frac{1}{n^2\pi^2}\left[\sin(n\pi x)\right]_{-1}^{1} \rightarrow B_n = \frac{2\cos(n\pi)}{n\pi}$$

This gives us the Fourier expansion

$$\boxed{1 + \sum_{n=1}^{\infty} \frac{2\cos(n\pi)}{n\pi} \sin(n\pi x)}$$

9. Find the matrix exponential using Putzer's Method for the following matrix A:

$$\begin{bmatrix} 9 & 8 \\ 6 & 7 \end{bmatrix}$$

Putzer's Method requires the eigenvalues of the matrix. These can be found with the equation $det(A - \lambda I) = 0$.

$$\begin{bmatrix} (9-\lambda) & 8 \\ 6 & (7-\lambda) \end{bmatrix}$$

$$(9-\lambda)\cdot(7-\lambda) - 48 = 0$$

$$\lambda^2 - 16 + 15 = 0 \Rightarrow (\lambda - 15)(\lambda - 1) = 0$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 15$.

Next, set up the corresponding B_0 and B_1 matrices:

$$B_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B_1 = A - \lambda_1 I = \begin{bmatrix} 8 & 8 \\ 6 & 6 \end{bmatrix}$$

Now, we must solve a series of first order ODEs to find r_1 and r_2 so Putzer's definition of the matrix exponential can be used.

(a) r_1 (Separable equation)

$$\frac{dr_1}{dt} = \lambda_1 r_1 \quad r_1(0) = 1$$

$$\frac{dr_1}{dt} = r_1 \to r_1 = Ce^t \to \boxed{r_1 = e^t}$$

(b) r_2 (Integrating factor)

$$\frac{dr_2}{dt} = \lambda_2 r_2 + r_1 \quad r_1(0) = 0$$

$$\frac{dr_2}{dt} = 15r_2 + e^t \to r_2 = Ce^{15t} - \frac{e^t}{14} \to \boxed{r_2 = \frac{e^{15t} - e^t}{14}}$$

Now use Putzer's definition of the matrix exponential:

$$e^{At} = B_0 \cdot r_1 + B_1 \cdot r_2$$

$$= B_0 \cdot (e^t) + B_1 \cdot (\frac{e^{15t} - e^t}{14})$$

$$= \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} + \begin{bmatrix} 8 \cdot \frac{e^{15t}-e^t}{14} & 8 \cdot \frac{e^{15t}-e^t}{14} \\ 6 \cdot \frac{e^{15t}-e^t}{14} & 6 \cdot \frac{e^{15t}-e^t}{14} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} \frac{8e^{15t}+6e^t}{14} & \frac{8e^{15t}-8e^t}{14} \\ \frac{6e^{15t}-6e^t}{14} & \frac{6e^{15t}+8e^t}{14} \end{bmatrix}$$