# MATH 241

## Midterm 4 Review

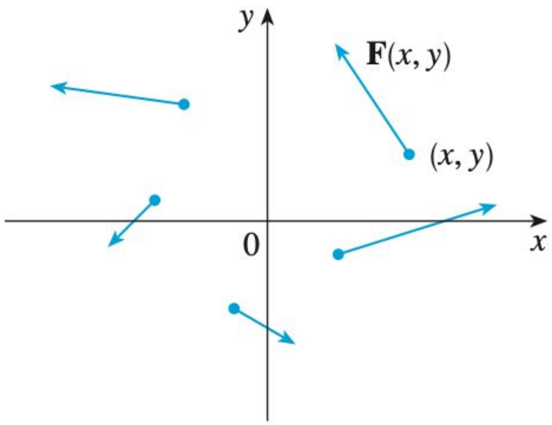
Keep in mind that this presentation was created by CARE tutors, and while it is thorough, it is not comprehensive.

# QR Code to the Queue

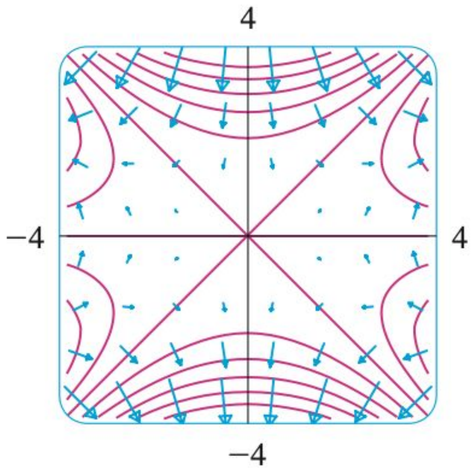
The queue contains the worksheet and the solution to this review session

# Vector Field, Gradient Vector Field

- A vector field F(x,y) = P **i** + Q **j** is a function that assigns each point (x,y) a 2D vector




- A gradient vector field ∇ F(x,y) is a vector field that is always **perpendicular to the contour map**



# Line Integral Along a Curve with respect to...

- Arc length (orientation does not matter, **integral of C = integral of –C**)

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

- x, y (orientation matters, **integral of C = –integral of –C**)

$$\int_C f(x, y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$$

$$\int_C f(x, y)\, dy = \int_a^b f(x(t), y(t))\, y'(t)\, dt$$

# Line Integral of Vector Fields

- Let F be a continuous vector field defined on a curve C given by a vector function r(t), a ≤ t ≤ b. Line integral of F along C (**Work done**) is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\, dx + Q\, dy + R\, dz$$

where $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$

# Fundamental Theorem of Line Integrals

- Let C be a smooth curve given by the vector function r(t), a ≤ t ≤ b. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then:

$$\int_C \nabla f \cdot dr = f[r(b)] - f[r(a)]$$

# Conservative Vector Field

- Line integrals of a conservative vector field are independent of path

$\int_C F \cdot dr$ is independent of path D if and only if

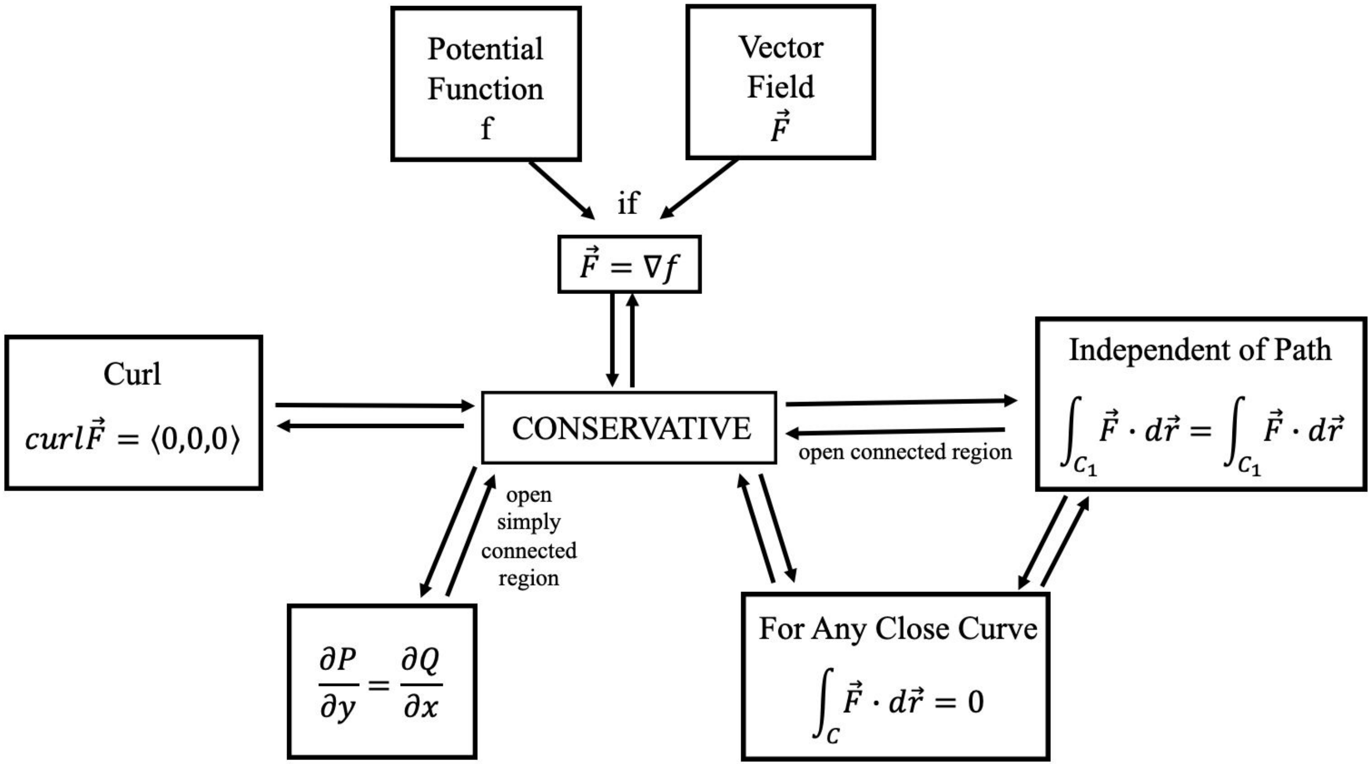
$\int_C F \cdot dr = 0$ for every closed path C in D

- Let F = P**i** + Q**j** be a vector field on an open simply-connected region D. Suppose that P and Q have continuous partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout } D$$

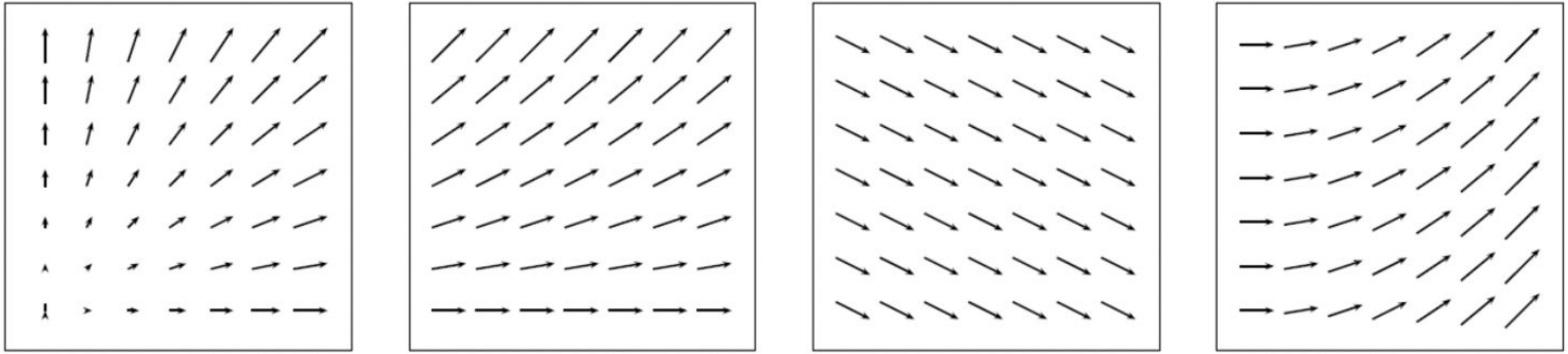
, then F is conservative.

# Conservative Vector Field

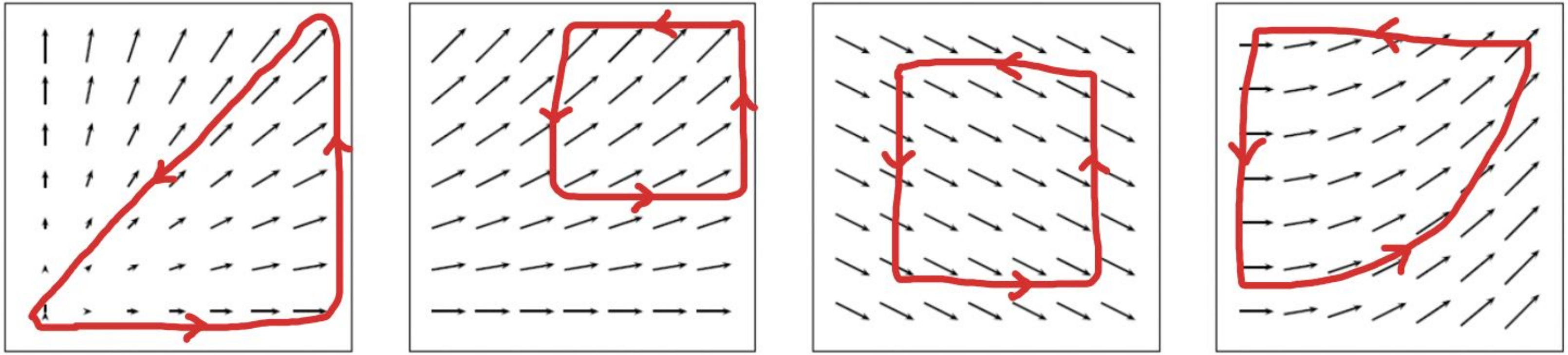


# Example Question #1

- **Which one of the vector fields shown below is not conservative?**



# Example Solution #1



**The fourth vector field is not conservative as line integral in the closed path does not equal to 0.**

# Green's Theorem

- Let C be a **counterclockwise**, **simple closed curve** in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

- Green's theorem to calculate the area of a region D bounded by C

$$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$$

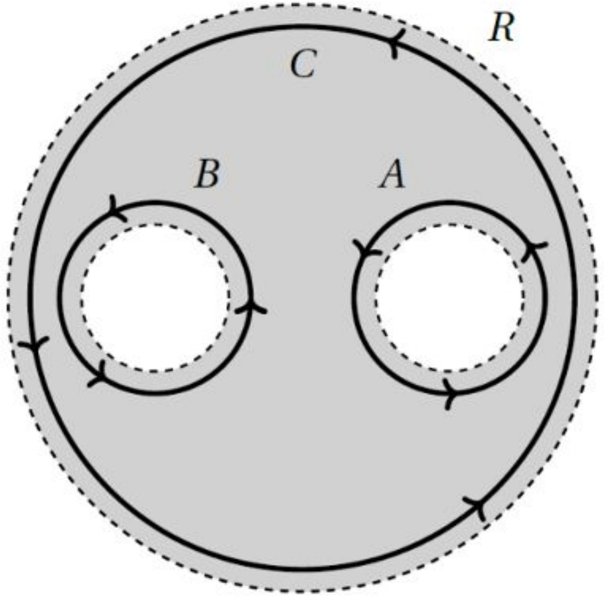
# Example Question #2

- Consider the region R shown at the right which contains simple closed curves A, B, and C. Suppose F = <P, Q> is a vector field with continuous partial derivatives on R with the following characteristics:

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \qquad \int_A F \cdot dr = 2 \qquad \int_B F \cdot dr = -1$$

(a) Find $\int_C F \cdot dr$

(b) Is this vector field conservative?

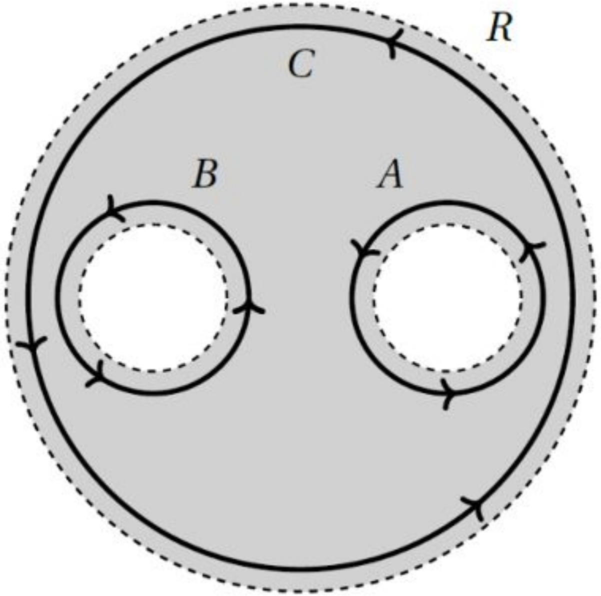


# Example Solution #2

(a) Let D be the region enclosed by C. Using Green's theorem:

$$\int_C F \cdot dr - \int_A F \cdot dr - \int_B F \cdot dr = 0$$

$$\int_C F \cdot dr - 2 - (-1) = 0 \qquad \boxed{\int_C F \cdot dr = 1}$$




(b) This vector field is not conservative because it is not a simply-connected region, and the line integral for the closed curve C is not 0.

# Curl

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F}$$

- Cross product → Curl is a **vector field**
- Describes how vectors **rotate** around a certain point
- Use **right-hand rule** to determine the sign of curl
- Curl of a gradient field = 0
- If F is conservative, curl = 0
- Green’s theorem in vector form:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl}\,\mathbf{F}) \cdot \mathbf{k}\, dA$$

# Curl Test for Conservative Vector Field

- If F is a vector field defined on all of $R^3$ whose component functions have **continuous partial derivatives** and **curl F = 0**, then F is a conservative vector field

# Divergence

$$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F}$$

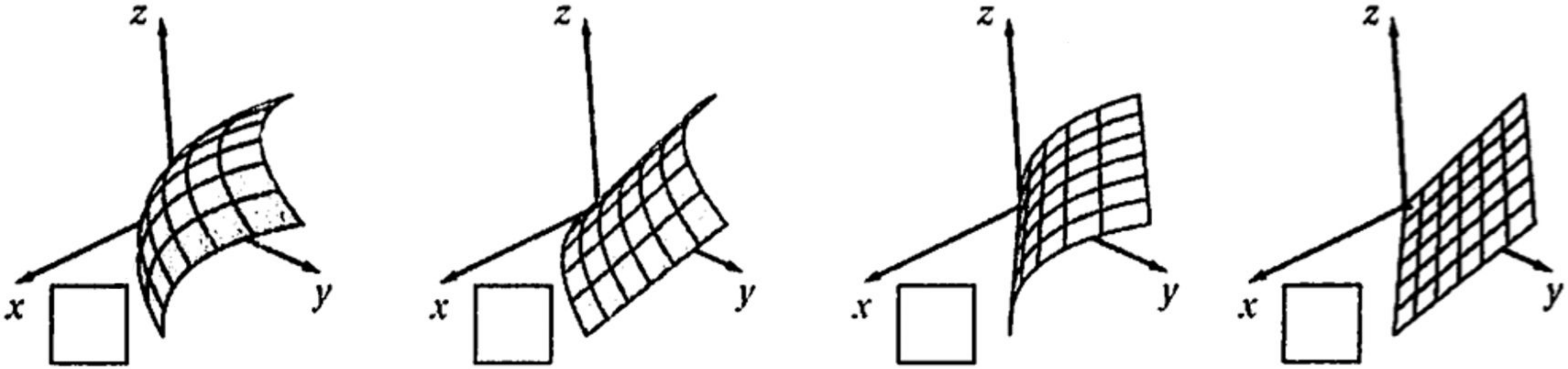
- Dot product → Divergence is a **scalar** field
- Describes how vectors diverge from a single point (or converge to a point)
- **Diverging vectors: positive, Converging vectors: negative**
- Green's theorem in vector form:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \text{div}\,\mathbf{F}(x, y)\, dA$$

# Example Problem #3

- **Match the surfaces below with the following parametrization:**

$$r(u, v) = \langle u, u^2 + v^2, v \rangle \text{ defined on } D = \{(u, v) | 0 \le u \le 1, 0 \le v \le 1\}$$



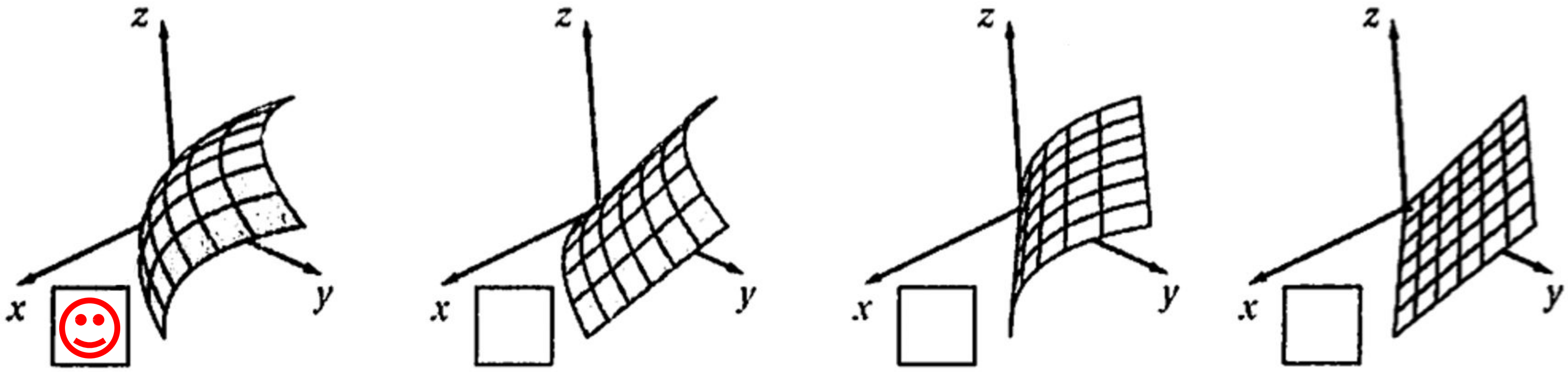
# Example Solution #3

$$r(u,v) = < u, u^2 + v^2, v > \text{ defined on } D = \{(u,v) | 0 \le u \le 1, 0 \le v \le 1\}$$

**When x is constant → curve on the yz-plane should be a parabola**

**When y is constant → curve on the xz-plane should be a circle**

**When z is constant → curve on the xy-plant should be a parabola**



# Surface Area of a Parametric Surface

- If a parametric surface S is given by the equation

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} \qquad (u, v) \in D$$

, the surface area of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

, where $r_u$ and $r_v$ are partial derivatives with respect to u and v.

# Surface Integral

- The surface integral of a function f over a parametric surface is:

$$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

# Flux

- The flux of a vector field F over a parametric surface is:

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_D \vec{F} \cdot (\vec{r_u} \times \vec{r_v})\, dA$$