Center for Academic Resources in Engineering (CARE)
Peer Exam Review Session

Math 220 – Calculus I

Midterm 3 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: April 23, 4pm-5:30pm Zac and Sophia

Session 2: April 24, 6pm-7:30pm Erik and Geo

Can't make it to a session? Here's our schedule by course:

https://care.grainger.illinois.edu/tutoring/schedule-by-subject

Solutions will be available on our website after the last review session that we host.

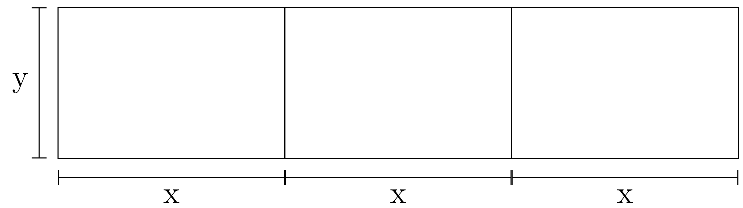
Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/844
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. A farmer wishes to fence off three identical adjoining rectangular pens as in the diagram shown, but only has 600 feet of fencing available. Determine the values for x and y which will maximize the total area enclosed by these three pens.



Area $= 3xy$
Total fencing $= 600 = 6x + 4y$

$$4y = 600 - 6x$$

$$y = 150 - 1.5x$$

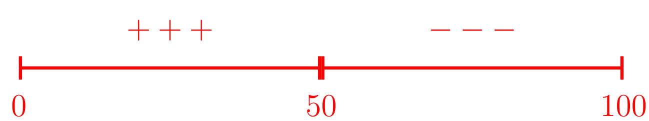
Area is now equivalent to:

$$\text{Area} = 3x * (150 - 1.5x)$$
$$\text{Area} = 450x - 4.5x^2$$

Now, we must maximize A for x in the range of $(0, 100)$

$$0 = \frac{dA}{dx}$$
$$0 = 450 - 9x$$
$$9x = 450$$
$$x = 50$$

Check for the values of A':



So we can see that there is an absolute maximum at $x = 50$. Evaluate y at $x = 50$

$$y = 150 - 1.5 * (50)$$
$$y = 75$$

$$\boxed{Area = 3 * 50 * 75 = 11,250 \text{ ft}^2}$$

2. Let R be the finite region bounded by the graphs of $y = 3\sin(x)$, $y = 6$, $x = 0$, and $x = \pi$. Set up, but do not evaluate, definite integrals which represent the following quantities. Integrate with respect to x.

(a) The area of the region, R.

(b) The volume of the solid formed when R is revolved around the line $y = 8$.

(c) The volume of the solid formed when R is revolved around the line $x = -2$.

(a)

$$A = \int_{x-min}^{x-max} (Y_{top} - Y_{bottom})dx = \boxed{\int_0^{\pi} (6 - 3\sin(x))dx}$$

(b)

$$V = \int_{x-min}^{x-max} (\text{cross-sectional area})dx = \int_0^{\pi} \pi((r_{out})^2 - (r_{in})^2)$$

$$\boxed{V = \int_0^{\pi} \pi((8 - 3\sin(x))^2 - (8-6)^2)dx}$$

(c)

$$V = \int_{x-min}^{x-max} (\text{surface area})dx = \int_0^{\pi} 2\pi * r * h dx = \boxed{\int_0^{\pi} 2\pi * (x+2) * (6 - 3\sin(x))dx}$$

3. Evaluate the following indefinite integral:

$$\int \frac{\sin^2(x)}{\sec(x)\csc^4(x)}dx$$

$$\int \frac{\sin^2(x)}{\frac{1}{\cos(x)} * \frac{1}{\sin^4(x)}}dx = \int \sin^6(x)\cos(x)dx$$

Need to use u-sub for this problem: $(u = \sin(x))$ and $(du = \cos(x)dx)$.

$$\int u^6 du = \frac{1}{7}u^7 + C = \boxed{\frac{1}{7} * \sin^7(x) + C}$$

4. Find the average value of the function below on the interval $[1, 9]$. Simplify.

$$f(x) = \frac{8x}{x^2+9}$$

Average Value

$$\left(\frac{1}{9-1}\right) * \int_1^9 \frac{8x}{(x^2+9)} dx$$

Need to use u-sub and set $u = x^2 + 9$ and $4du = 8xdx$. Remember to replace the limits of integration, as we are now integrating with respect to u rather than x.

$$\frac{1}{8}\int_{10}^{90} \left(\frac{4}{u}\right) du = \frac{1}{2}\big(\ln(90) - \ln(10)\big) = \boxed{\frac{1}{2}\ln(9)}$$

5. Evaluate the indefinite integral:

$$\int \frac{e^{9x}}{e^{18x}+1} dx$$

Need to use u-sub to solve this problem: $u = e^{9x}$ and

$$\frac{1}{9} du = e^{9x} dx$$

$$\int \frac{1}{u^2+1} * \frac{1}{9} du = \frac{1}{9}\big(\arctan(u)\big) + C = \boxed{\frac{1}{9}\big(\arctan(e^{9x})\big) + C}$$

6. At t hours, a population of bacteria is growing at a rate of

$$r(t) = \frac{21e^{\sqrt{t}}}{\sqrt{t}} \text{ bacteria per hour}$$

Compute the change in population size between times $t = 169$ s and $t = 225$ s. Simplify your answer.

Net change in population from t = 169 to t = 225 is defined as:

$$\int_{169}^{225} r(t) dt$$

$$= \int_{169}^{225} \frac{(21 * e^{t^{\frac{1}{2}}})}{t^{\frac{1}{2}}} dt$$

$\rightarrow$ u-sub $u = t^{\frac{1}{2}}$ and $2du = (\frac{1}{t^{\frac{1}{2}}})dt$ and $t = 169$ equates to $u = 13$ and $t = 225$ equates to $u = 15$

$$\int_{13}^{15} (21 * 2 * e^u)du = \boxed{42 * e^{15} - 42 * e^{13} \text{ bacteria}}$$

7. The function $f(x) = 10x^3 - 20x + 1$ has one root in the interval $[1, 2]$. In order to approximate this root, begin with an initial estimate of $x_1 = 2$ and use Newton's Method to obtain a second estimate x_2, then write it in decimal form.

$$f'(x) = 30x^2 - 20$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 2 - \frac{f(2)}{f'(2)}$$

$$x_2 = 2 - \frac{41}{100} = 2 - 0.41 = \boxed{1.59}$$

8. Express the definite integral as the limit of Riemann Sums. Do not evaluate the limit.

$$\int_{-3}^{5} x^2 e^{sin(x)} dx$$

$$f(x) = x^2 * e^{\sin(x)}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\delta x \to \lim_{n\to\infty} \sum_{k=1}^{n} (x_k)^2(\delta x)(e^{\sin(x)})$$

$$\delta x = \frac{(b-a)}{n} = \frac{8}{n}$$

$$x = a + k * (\delta x) = -3 + \left(\frac{8}{n}\right)k$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(-3 + \left(\frac{8k}{n}\right)\right)^2 \left(e^{\sin\left(-3+\frac{8k}{n}\right)}\right)\left(\frac{8}{n}\right)$$

9. Evaluate the definite integral:

$$\int_{1}^{3} \frac{12x + 18}{x^2 + 3x + 10} dx$$

We can use u-sub to solve this.

$$\int_{1}^{3} \frac{6(2x + 3)}{x^2 + 3x + 10} dx$$

If we consider $u = x^2 + 3x + 10$, that requires that $du = (2x + 3)dx$, which we see in the numerator.

$$\int_{14}^{28} \frac{6du}{u}$$

Notice that the bounds have changed; they were previously $x = 1$ and $x = 3$, but we used the definition of u to redefine them as $u = 1^2 + 3(1) + 10 = 14$ and $u = 3^2 + 3(3) + 10 = 28$, respectively.

$$6 \int_{14}^{28} \frac{1}{u} du$$

$$6 \ln |u| \Big|_{14}^{28}$$

$$6(\ln(28) - \ln(14))$$

$$6 \ln\left(\frac{28}{14}\right)$$

$$\boxed{6\ln(2)}$$

10. Some of the values of a polynomial $f(x)$ are shown below in the table. If $g(x) = 8xf'(x^2)$, then find the average value of $g(x)$ on the interval $[0, 2]$. Simplify your answer.

x	$f(x)$
0	3
1	5
2	8
3	13
4	21
5	34
6	55
7	89
8	144
9	233

Average-value

$$\left(\frac{1}{2-0}\right)\int_0^2 g(x)dx = \frac{1}{2}\int_0^2 8x * f'(x^2)dx$$

$\rightarrow$ use u-sub where $u = x^2$ and $4du = 8xdx$

Average-value

$$\frac{1}{2}\int_0^4 4f'(u)du = 2\big(f(4) - f(0)\big) = \boxed{36}$$

11. Check if the Fundamental Theorem of Calculus applies, then evaluate the given integral if it does.

(a)
$$\frac{d}{dx}\int_{-\pi}^{x} \sin(t)dt, \qquad x \in [-\pi, \pi]$$

(b)
$$\frac{d}{dx}\int_{-5}^{x} \frac{1}{t}dt, \qquad x \in [-5, 3]$$

(a) This takes the form $\frac{d}{dx}\int_a^x f(t)dt$ where f is continuous on the given interval. We therefore can say immediately that this evaluates to $f(x)$, or in this case, $\boxed{\sin(x)}$ on this interval.

Alternatively, one can evaluate the integral as follows:

$$\begin{aligned}\frac{d}{dx}\int_{-\pi}^{x} \sin(t)dt &= \frac{d}{dx}(-\cos(x) - (-\cos(\pi))\\ &= \frac{d}{dx}(-\cos(x) - 1)\\ &= \boxed{\sin(x)}\end{aligned}$$

(b) This appears to take a similar form as part (a), but it is important to notice that the given function $1/x$ is *not* continuous on the given interval (it has a discontinuity at $x = 0$). Therefore, the FTC does not apply to this problem. It *would* apply on an interval such as $[-5, -1]$.