# Center for Academic Resources in Engineering (CARE)
# Peer Exam Review Session

**Math 241 – Calculus III**

**Midterm 3 Worksheet Solutions**

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*The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.*

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Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Apr. 3, 4 - 6pm Gabe, Camila in CIF 2039

Session 2: Apr. 4, 7-9pm Matthew, Pallab in CIF 3018

Can't make it to a session? Here's our schedule by course:

https://care.grainger.illinois.edu/tutoring/schedule-by-subject

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/845
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

**Please be sure to follow the above steps to add yourself to the Queue.**

Good luck with your exam!

1. Compute the double integral over the indicated rectangle. Confirm your answer by switching the order of integration and recomputing.

$$\iint_R 2x - 4y^3 \; dA \qquad R = [-5, 4] \times [0, 3]$$

$$\int_0^3 \int_{-5}^4 2x - 4y^3 \; dxdy$$
$$\int_0^3 -9 - 36y^3 \; dy$$
$$-9y - 9y^4\Big|_0^3 = \boxed{-756}$$

$$\int_{-5}^4 \int_0^3 2x - 4y^3 \; dydx$$
$$\int_{-5}^4 6x - 81 \; dx$$
$$3x^2 - 81x\Big|_{-5}^4 = \boxed{-756}$$

2. Making an appropriate change of variables, compute the following double integral over the region bound by a circle of radius 2 and a circle of radius 5.

$$\iint_D e^{x^2+y^2} \; dA$$

Using polar coordinates gives the following integral

$$\iint_D e^{x^2+y^2} \; dA = \int_0^{2\pi} \int_2^5 re^{r^2} \; drd\theta$$

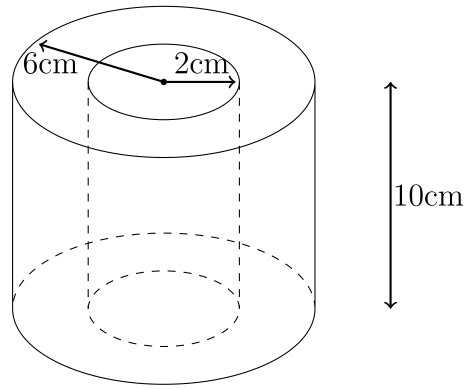
Then using a $u-$substitution $u = r^2$ for the $r$ dependence

$$\frac{1}{2}\int_0^{2\pi} \int_4^{25} e^u \; dud\theta = \boxed{\pi(e^{25} - e^4) \approx 2.26 \times 10^{11}}$$

3. A toilet paper manufacturing company has increased their production. Unfortunately, this production increase has caused a major manufacturing error! As you move towards the center of any one toilet paper roll, the sheets get progressively more dense. The density of a toilet paper roll can be modeled using the following function:

$$\rho(r) = \cos\left(\frac{\pi(r-1)}{6}\right) + 1$$

$r$ is the radial distance away from the center of the roll (inside the center cardboard tube). The whole roll can be modeled as a cylinder with an outer radius of 6, and inner radius of 2 (the cardboard tube radius), and a height of 10.




(a) Without using a calculator, calculate the mass of the toilet paper roll if the density everywhere was just 1 (leave $\pi$ in your answer)

(b) Set up the triple integral to solve for the mass of a toilet paper roll. Neglect the weight of the inner cardboard tube for your calculation.

(c) Without using a calculator, solve the integral (leave $\pi$ in your answer)

(a) The mass of the tube is equal to the density times the volume:

$$m = \rho V = \rho\pi(r_{outer}^2 - r_{inner}^2)h = \boxed{320\pi}$$

(b) Cylindrical coordinates are most useful here since the figure is a cylinder. The mass is found by integrating the density with respect to d$V$ (in cylindrical coordinates).

$$\boxed{\int_0^{10}\int_0^{2\pi}\int_2^6 \Big(\cos\big(\frac{\pi(r-1)}{6}\big) + 1\Big) r\ dr d\theta dz}$$

(c)

$$\int_0^{10}\int_0^{2\pi}\int_2^6 \Big(\cos\big(\frac{\pi(r-1)}{6}\big) + 1\Big) r dr d\theta dz$$

$$(10)(2\pi)\int_2^6 \Big(\cos\big(\frac{\pi(r-1)}{6}\big)+1)\Big) r dr$$

$$20\pi\Big[\Big(\frac{r^2}{2}\Big)\Big|_2^6 + \int_2^6 \cos\big(\frac{\pi(r-1)}{6}\big) r dr\Big]$$

Integration by parts

$$u = r, v = \frac{6}{\pi}\sin\left(\frac{\pi}{6}(r-1)\right)$$

$$20\pi\left[18 - 2 + \left(\frac{6}{\pi}\sin\left(\frac{\pi}{6}(r-1)\right)r\right)_2^6 - \int_2^6 \frac{6}{\pi}\sin\left(\frac{\pi}{6}(r-1)\right)dr\right]$$

$$20\pi\left[16 + \frac{36}{\pi}\sin\left(\frac{5\pi}{6}\right) - \frac{12}{\pi}\sin\left(\frac{\pi}{6}\right) - \int_2^6 \frac{6}{\pi}\sin\left(\frac{\pi}{6}(r-1)\right)dr\right]$$

$$20\pi\left[16 + \frac{18}{\pi} - \frac{6}{\pi} + \frac{6^2}{\pi^2}\cos\left(\frac{\pi}{6}(r-1)\right)\Big|_2^6\right]$$

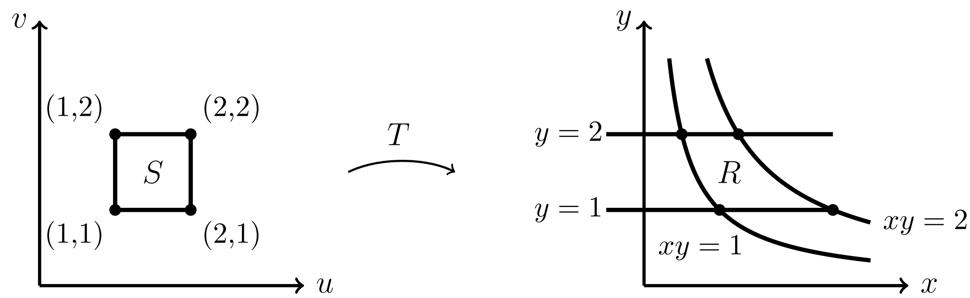
$$20\pi\left[16 + \frac{12}{\pi} + \frac{36}{\pi^2}\left(\cos\left(\frac{5\pi}{6}\right) - \cos\left(\frac{\pi}{6}\right)\right)\right]$$

$$20\pi\left[16 + \frac{12}{\pi} + \frac{36}{\pi^2}\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)\right]$$

$$20\pi\left[16 + \frac{12}{\pi} - \frac{36\sqrt{3}}{\pi^2}\right]$$

$$\boxed{320\pi + 240 - \frac{720\sqrt{3}}{\pi}}$$

4. Consider the region $R$:




(a) Suppose there exists a transformation $S$: $\mathbb{R}^2 \to \mathbb{R}^2$ from $S$ to $R$. Find $T(u, v)$

(b) Use the answer from (a) to evaluate $\iint_R x^2 \mathrm{d}A$

(a) It's convenient to make the substitution $v = y$ since both v and y have common lines at $v = y = 1$ and $v = y = 2$. The substitution of u would then be $u = xy$. (Check $u = 1$ and $u = 2$. They match to $xy = 1$ and $xy = 2$.)

$$v = y \in [1, 2]$$

$$u = xy \in [1, 2]$$

To solve for $x$, divide by $y$ and then substitute in $v$ for $y$.

$$(x, y) = T(u, v) = \left(\frac{u}{v}, v\right)$$

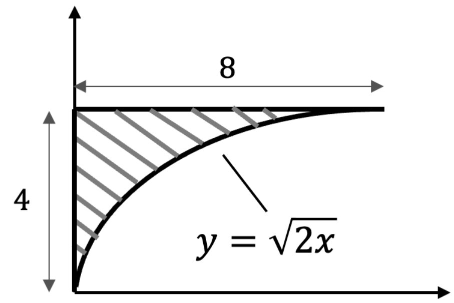
(b) Substitute $\frac{u}{v}$ for $x$ and $v$ for $y$ calculate the Jacobian for this transformation.

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$

Note that the Jacobian matrix is taken as the absolute value of the determinant. Now calculate the integral

$$\int_1^2 \int_1^2 \left(\frac{u}{v}\right)^2 \frac{1}{v} du dv = \boxed{\frac{7}{8}}$$

5. What is the x-coordinate of the center of mass for the shaded region if it has a density function $\rho(x, y) = 3x+2y$? (Solve the integral by hand then evaluate the final expression with a calculator.)




$$\overline{x} = \frac{1}{m}\iint x\rho(x, y)dA, m = \iint \rho(x, y)dA$$

$$y = \sqrt{2x} \rightarrow x = \frac{y^2}{2}$$

$$\int_0^4 \int_0^{y^2/2} x(3x + 2y)dxdy = \int_0^4 \frac{y^6}{8} + \frac{y^5}{4}dy = \frac{y^7}{56} + \frac{y^6}{24}\bigg|_0^4 \approx 463.238$$

$$m = \int_0^4 \int_0^{y^2/2} (3x + 2y)dxdy = \int_0^4 \frac{3y^4}{8} + y^3dy = \frac{3y^5}{40} + \frac{y^4}{4}\bigg|_0^4 = 140.8$$

$$\overline{x} = \frac{463.238}{140.8} \approx \boxed{3.29}$$

6. Find the surface area of $z = \frac{x^2+y^2}{2}$ that lies within the cylinder $x^2 + y^2 = 4$. (Evaluate the integral by hand.)

$$S = \iint_D \sqrt{(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 + 1} dA$$

$$\frac{\partial z}{\partial x} = x, \frac{\partial z}{\partial y} = y$$

$$S = \iint_D \sqrt{x^2 + y^2 + 1} dA$$

Use Polar Coordinates.

$$S = \iint_D \sqrt{r^2 + 1} r dr d\theta$$

Use $u = r^2 + 1$ and $du = 2rdr$

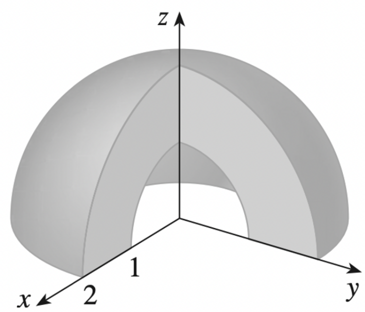
$$S = \iint_D \frac{\sqrt{u}}{2} du d\theta = \frac{1}{3} u^{3/2} \int_\theta d\theta = \frac{1}{3}(r^2+1)^{3/2}\Big|_0^2 \int_0^{2\pi} d\theta$$

(The upper bound for r is 2 because $r^2 = 4$ from the cylinder)

$$S = \boxed{\frac{1}{3}(\sqrt{125} - 1) \cdot 2\pi}$$

7. Set up the triple integral of the function $f(x, y)$ using spherical coordinate over the solid shown below.

$$f(x,y) = \frac{x^2+y^2}{z^2}$$




The bounds of the solid are shown below:

$$\rho : 1 \le \rho \le 2$$

$$\theta : \frac{\pi}{2} \le \theta \le 2\pi$$

$$\phi : 0 \le \phi \le \frac{\pi}{2}$$

Convert $f(x, y, z)$ into $f(\rho, \theta, \phi)$

$$\frac{x^2+y^2}{z^2} = \frac{(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2}{(\rho\cos\phi)^2} = \tan^2\phi$$

$$\iiint_E f(x,y,z)dV = \boxed{\int_0^{\pi/2}\int_{\pi/2}^{2\pi}\int_1^2 (\tan^2\phi)(\rho^2\sin\phi)d\rho d\theta d\phi}$$

8. Using cylindrical coordinate to set up the integral to calculate the mass of a solid that is enclosed by both the cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2 = 8$. The density of the solid is modeled by $f(x, y, z) = \arctan(y/x)$.
Use cylindrical coordinates to represent $z$ as a function of $r$.
From the cone equation: $z = \sqrt{r^2} = r$
From the sphere equation: $z^2 = 8 - (x^2+y^2) = 8 - r^2 \to z = \sqrt{8-r^2}$
Therefore, $r \le z \le \sqrt{8-r^2}$
$\theta$ spans the entire xy-plane, so $0 \le \theta \le 2\pi$
To solve for the bounds of $r$, equate the cone and sphere equations.
$r = \sqrt{8-r^2} \to 8 - r^2 = r^2 \to r^2 = 4 \to r = 2$ (Note that there is no negative radius, so $-2$ is omitted.)
The function becomes $\arctan(y/x) = \arctan(\tan\theta) = \theta$
$\boxed{\int_0^2 \int_0^{2\pi} \int_r^{\sqrt{8-r^2}} \theta r dz d\theta dr}$

9. Evaluate $\iint_R 6x - 3y dA$ where $R$ is the parallelogram with vertices $(2, 0)$, $(5, 3)$, $(6, 7)$, and $(3, 4)$ using the transformation $x = \frac{v-u}{3}$ and $y = \frac{4v-u}{3}$.

Plot the vertices in the xy-plane in the uv-plane using the transformation equations. First by rearranging the equation into $u = f(x, y)$ and $v = f(x, y)$

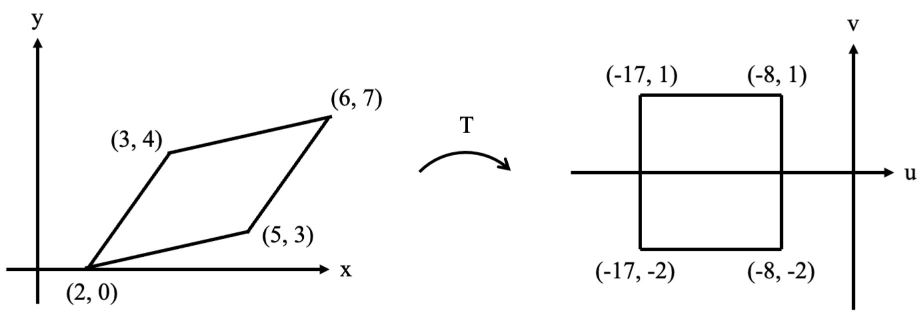
$3x = v - u$, $3y = 4v - u$

Combining the two equations to eliminate u , $3x - 3y = -3v$

$v = y - x$, plug this back into $3x = v - u$

$3x = y - x - u \to u = y - 4x$

Plugging in the x,y coordinates to $u = y - 4x$, $v = y - x$




The bounds of u,v are then $-17 \leq u \leq -8$, $-2 \leq v \leq 1$. Convert the function $6x - 3y$ into $f(u, v)$:

$$f(u, v) = 6(\frac{v-u}{3}) - 3(\frac{4v-u}{3}) = -2v - u$$

Then, we calculate the absolute value of the Jacobian:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{4}{3} \end{vmatrix} = \frac{1}{3}$$

$$\iint_R 6x - 3y dA = \frac{1}{3} \int_{-17}^{-8} \int_{-2}^{1} (-2v - u) dv du = \frac{1}{3} \int_{-17}^{-8} (3 - 3u) du = \boxed{\frac{243}{2}}$$