ILLINOIS

Center for Academic Resources in Engineering (CARE)
Peer Exam Review Session

Math 231 – Calculus II

Midterm 3 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Mar. 28, 5-7pm Grace and Pranav

Session 2: Mar. 31, 6-8pm Soundarya and Bella

Can't make it to a session? Here's our schedule by course:

`https://care.grainger.illinois.edu/tutoring/schedule-by-subject`

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/844
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. Determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{n^2+3}{n^3+3}$$

$\frac{n^2+3}{n^3+3}$ can be compared to $\frac{n^2}{n^3}$ by the comparison test.

$\frac{n^2}{n^3} = \frac{1}{n}$ which diverges by the p-test (p =1).

Because $\frac{n^2+3}{n^3+3} > \frac{n^2}{n^3}$, the series $\boxed{\text{diverges}}$.

2. Determine whether the series converges or diverges. Note that $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$.

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

Use the divergence test: $a_n = \frac{n!}{n^n}$ and $a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

$\frac{a_{n+1}}{a_n} = [\frac{(n+1)!}{(n+1)^{n+1}}][\frac{n^n}{n!}] = \frac{n^n}{(n+1)^n} = (\frac{n}{n+1})^n$

$\lim_{n\to\infty}(\frac{n}{n+1})^n = \lim_{n\to\infty}(\frac{1}{1+\frac{1}{n}})^n = \frac{1}{e} < 1$

So the series $\boxed{\text{converges}}$.

3. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{5^n}{n+6^n}$$

$\frac{5^n}{n+6^n} < \frac{5^n}{6^n} = (\frac{5}{6})^n$ which converges. Thus, $\frac{5^n}{n+6^n}$ $\boxed{\text{converges}}$.

4. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} (\frac{n^2}{2n^2+5})^n$$

Use the root test:

$(\frac{n^2}{2n^2+5})^n \to [a_n]^{\frac{1}{n}} \to [(\frac{n^2}{2n^2+5})^n]^{\frac{1}{n}}$

$= \frac{n^2}{2n^2+5}$

$\lim_{n\to\infty} \frac{n^2}{2n^2+5} = \frac{1}{2} < 1 \to \boxed{\text{Series converges}}$.

5. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{17n^2}{3n^4-1}$$

Use the limit comparison test:

$a_n = \frac{17n^2}{3n^4-1}$ and $b_n = \frac{1}{n^2}$

$\frac{a_n}{b_n} = (\frac{17n^2}{3n^4-1})(\frac{n^2}{1}) = \frac{17}{3} \to$ both do the same thing.

$\frac{1}{n^2}$ converges by the p-test $\to \boxed{\text{both converge}}$.

6. Determine if the following series converges absolutely, converges conditionally, or diverges.

$$\sum_{n=1}^{\infty} \frac{(2n+1)(-2)^n}{n!}$$

$$a_n = \frac{(2n+1)(-2)^n}{n!}, \qquad a_{n+1} = \frac{(2n+3)(-2)^{n+1}}{(n+1)!}$$

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{-2(2n+3)n!}{(n+1)!(2n+1)}\right| \\
&= \lim_{n\to\infty} \left|\frac{-2(2n+3)}{(n+1)(2n+1)}\right| \\
&= \lim_{n\to\infty} \left|\frac{-4n-6}{2n^2+...}\right| \\
&= \lim_{n\to\infty} = 0 < 1
\end{aligned}$$

Therefore the series $\boxed{\text{converges absolutely.}}$

7. You are given the power series

$$\sum_{n=1}^{\infty} \frac{(x-7)^n}{3^n(2n+1)}$$

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(a) Find the radius of convergence.

(b) Find the interval of convergence.

(a)

$$a_n = \frac{(x-7)^n}{3^n(2n+1)}, \qquad a_{n+1} = \frac{(x-7)^{n+1}}{3^{n+1}(2n+3)}$$

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x-7)^{n+1}}{3^{n+1}(2n+3)}\frac{3^n(2n+1)}{(x-7)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{2n+1}{2n+3}\frac{x-7}{3}\right| \\
&= \left|\frac{x-7}{3}\right| < 1 \\
&\implies |x-7| < 3 \implies \boxed{R = 3}
\end{aligned}$$

(b)

$$-3 < x-7 < 3$$
$$4 < x < 10$$

$$\text{If } x = 4, \quad \sum_{n=1}^{\infty} \frac{(4-7)^n}{3^n(2n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{3^n(2n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$$

This series at $x = 4$ converges by the alternating series test.

$$\text{If } x = 10, \quad \sum_{n=1}^{\infty} \frac{(10-7)^n}{3^n(2n+1)} = \sum_{n=1}^{\infty} \frac{1}{2n+1}$$

This series at $x = 10$ diverges by the limit comparison test with $\frac{1}{n}$.
Therefore, the interval of convergence is $\boxed{[4, 10)}$.

8. Let $a_n = n\left(\frac{x+3}{2}\right)^n$ and $b_n = \frac{1}{n^2+1}\left(\frac{x-1}{5}\right)^n$. Let $f(x) = \sum_{n=0}^{\infty} a_n$ and $g(x) = \sum_{n=0}^{\infty} b_n$.

(a) Find the IOC for f and g. Where do these converge absolutely?

(b) Carefully find the IOC for $h(x) = \sum_{n=0}^{\infty}(a_n + b_n)$.

(a) Use the Root Test on f:

$$L = \lim_{n\to\infty}\left|n\left(\frac{x+3}{2}\right)^n\right|^{\frac{1}{n}} = \left|\frac{x+3}{2}\right|\lim_{n\to\infty} n^{\frac{1}{n}} = \left|\frac{x+3}{2}\right|$$

The series will converge absolutely when $L < 1$:

$$\left|\frac{x+3}{2}\right| < 1 \implies -1 < \frac{x+3}{2} < 1 \implies -2 < x+3 < 2 \implies -5 < x < -1$$

So the series converges absolutely on $(-5,-1)$. We also check the endpoints:

$$x=-5 : \sum_{n=0}^{\infty} n\left(\frac{-5+3}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^n n \text{ which diverges by Divergence Test}$$
$$x=-1 : \sum_{n=0}^{\infty} n\left(\frac{-1+3}{2}\right)^n = \sum_{n=0}^{\infty} n \text{ which also diverges by Divergence Test}$$

So $\boxed{f \text{ converges absolutely on } (-5,-1) \text{ and diverges everywhere else}}$. What about g? Use the Root Test again:

$$L = \lim_{n\to\infty}\left|\frac{1}{n^2+1}\left(\frac{x-1}{5}\right)^n\right|^{\frac{1}{n}} = \left|\frac{x-1}{5}\right|\lim_{n\to\infty}\left(\frac{1}{n^2+1}\right)^{\frac{1}{n}} = \left|\frac{x-1}{5}\right|$$

Again, we need $L < 1$ for absolute convergence:

$$\left|\frac{x-1}{5}\right| < 1 \implies -1 < \frac{x-1}{5} < 1 \implies -5 < x-1 < 5 \implies -4 < x < 6$$

So g converges absolutely on $(-4,6)$. What about the endpoints?

$$x=-4 : \sum_{n=0}^{\infty}\frac{1}{n^2+1}\left(\frac{-4-1}{5}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+1} \text{ which converges absolutely}$$
$$x=6 : \sum_{n=0}^{\infty}\frac{1}{n^2+1}\left(\frac{6-1}{5}\right)^n = \sum_{n=0}^{\infty}\frac{1}{n^2+1} \text{ which also converges absolutely}$$

So $\boxed{g \text{ converges absolutely on } [-4,6] \text{ and diverges everywhere else}}$.

(b) The termwise addition of series is allowed only where both series converge absolutely. Hence, h converges on the intersection of the intervals on which f and g converge absolutely: This is $\boxed{(-5,-1)\cap[-4,6] = [-4,-1)}$.

9. Determine whether the series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Using the integral test:

$a_n = \frac{1}{n \ln n}$

$\int_2^\infty \frac{1}{x \ln x} dx$; Using u-substitution to solve the integral: $u = \ln x, du = \frac{1}{x} dx$

$\int_2^\infty \frac{1}{x \ln x} dx = \int_2^\infty \frac{1}{u} du = \lim_{t \to \infty} \ln(\ln t) - \ln(\ln 2) = \infty$

$\frac{1}{n \ln n}$ diverges by the integral test $\rightarrow$ $\boxed{\text{The series diverges}}$.