Center for Academic Resources in Engineering (CARE)
Peer Exam Review Session

Math 220/221 – Calculus I

Midterm 2 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: March 26, 4:30-6:00 pm Zac and Anthony

Session 2: March 27, 6:00-7:30 pm Pavle and Sophia

Can't make it to a session? Here's our schedule by course:

`https://care.grainger.illinois.edu/tutoring/schedule-by-subject`

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/844
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. Find $f'(x)$ given that $f(x) = \sqrt[4]{\arctan(x^9)}$

$$f(x) = \sqrt[4]{\arctan(x^9)}$$

$$f(x) = \left(\arctan\left(x^9\right)\right)^{\frac{1}{4}}$$

$$f'(x) = \frac{1}{4}\left(\arctan\left(x^9\right)\right)^{-\frac{3}{4}} * \frac{d}{dx}\left(\arctan\left(x^9\right)\right)$$

$$f'(x) = \frac{1}{4}\left(\arctan\left(x^9\right)\right)^{-\frac{3}{4}} * \left(\frac{1}{x^{18}+1}\right) * \frac{d}{dx}(x^9)$$

$$\boxed{f'(x) = \frac{1}{4}\left(\arctan\left(x^9\right)\right)^{-\frac{3}{4}} * \left(\frac{1}{x^{18}+1}\right) * 9x^8}$$

2. Suppose that A represents the number of grams of a radioactive substance at time t seconds. Given that $\frac{dA}{dt} = -0.125A$, how long does it take 12 grams of the substance to be reduced to 4 grams?

First recall that $\frac{d}{dx} = ky$ so $y = ce^{kx}$. So $\frac{dA}{dt} = -0.125A$ and $A = ce^{-0.125t}$

Plugging in $A = 12$ when $t = 0$ gives us $12 = ce^{-0.125*0}$ gives us $12 = c$. Thus, $A = 12e^{-0.125t}$

$$4 = 12e^{-0.125t}$$

$$\ln\left(\frac{4}{12}\right) = -0.125t$$

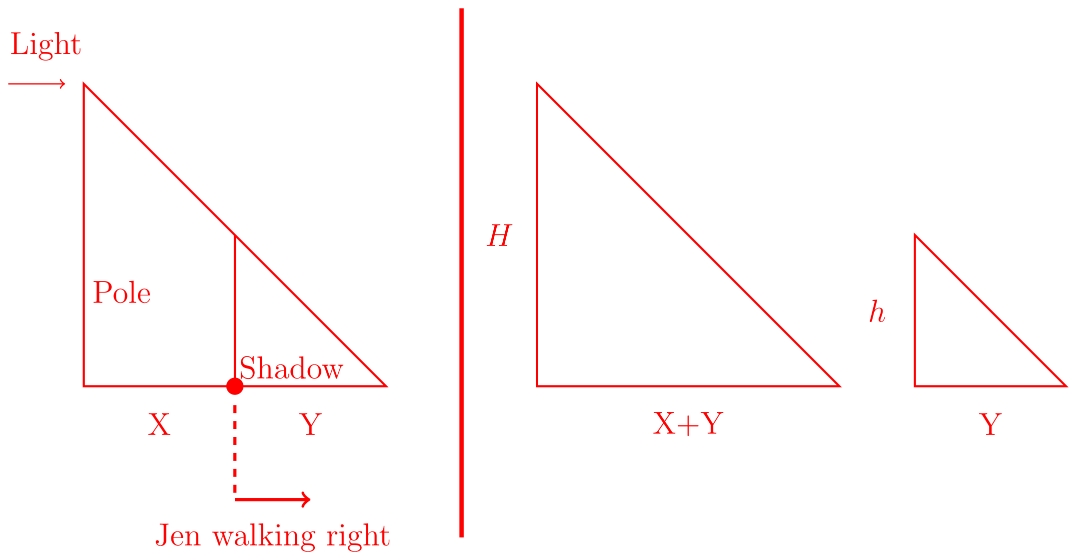
$$t = \frac{\ln\left(\frac{1}{3}\right)}{-0.125} = 8\ln(3)$$

$$t = \boxed{8\ln(3)\text{ s}}$$

3. A streetlight is mounted at the top of a tall pole with $H = 16.5$ ft. Jennifer's height is $h = 5.5$ ft tall. She walks away from the pole with a speed of 8 ft/s along a straight path. how quickly is the length of her shadow on the ground increasing when she is 15 ft from the pole?

Given: $\frac{dx}{dt} = 8$, we want $\frac{dy}{dx}|_{x=15}$

Use the below diagrams to help solve the problem



$$\frac{X+Y}{H} = \frac{Y}{h} = X+Y = \frac{Y}{h} * H = 3Y$$

$$X = 2Y$$

$$\frac{d}{dt}(X) = \frac{d}{dt}(2Y)$$

$$\frac{dx}{dt} = 2\frac{dy}{dt}$$

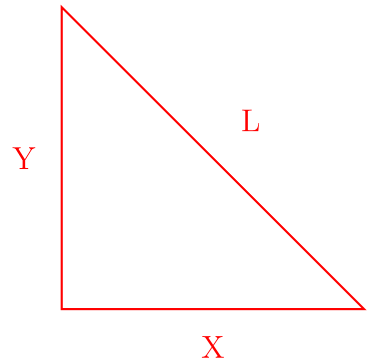
$$8 = 2\frac{dy}{dt}$$

$$\frac{dy}{dt} = 4$$

Therefore, the shadow length is increasing at a rate of $\boxed{\text{4 ft/s}}$.

4. The top of a ladder slides down a vertical wall at a rate of 8 m/s. At the moment when the bottom of the ladder is 4 meters from the wall, it slides away from the wall at a rate of 15 m/s. How long is the ladder?

Note that L is a constant length.



Given: $\frac{dy}{dt} = -8$ and $\frac{dx}{dt}|_{x=4} = 15$

We want L:

$$X^2 + Y^2 = L^2$$

$$\frac{d}{dt}(X^2 + Y^2) = \frac{d}{dt}(L^2)$$

$$2X * \frac{dx}{dt} + 2Y * \frac{dy}{dt} = 0$$

$$2 * 4 * 15 + 2 * Y * (-8) = 0$$

$$Y = \frac{15}{2}$$

$$L = \sqrt{X^2 + Y^2}$$

$$\boxed{L = 8.5 \text{ m}}$$

5. Find the absolute minimum y-value of the given function:

$$y = \frac{2x}{\sqrt{x - 81}}$$

Domain of the function: $x > 81$.

$$y' = \frac{2 * \sqrt{x-81} - (2x) * (\frac{1}{2}) * (x-81)^{-\frac{1}{2}}}{(\sqrt{x-81})^2}$$

$$y' = \frac{2\sqrt{x-81} - \frac{x}{\sqrt{x-81}}}{x - 81}$$

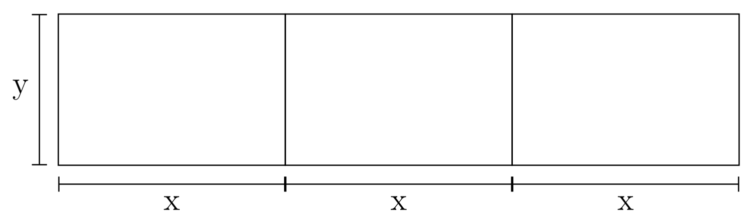
$$y' = \frac{2(x-81) - x}{(x-81)(\sqrt{x-81})}$$

$$y' = \frac{x-162}{(x-81)\sqrt{x-81}}$$

There is an absolute min at $x = 162$ because that is where y' equals 0. The min is $\boxed{36}$, which can be found by plugging 162 into the function for y.

Check limits towards infinity for proper absolute minimum. For this function, as x approaches infinity, limit is infinity, hence no absolute minimum. Domain excludes negative infinity, so limit towards negative infinity DNE.

6. A farmer wishes to fence off three identical adjoining rectangular pens as in the diagram shown, but only has 600 feet of fencing available. Determine the values for x and y which will maximize the total area enclosed by these three pens.



Area $= 3xy$
Total fencing $= 600 = 6x + 4y$

$$4y = 600 - 6x$$

$$y = 150 - 1.5x$$

Area is now equivalent to:

$$\text{Area} = 3x * (150 - 1.5x)$$
$$\text{Area} = 450x - 4.5x^2$$

Now, we must maximize A for x in the range of $(0, 100)$

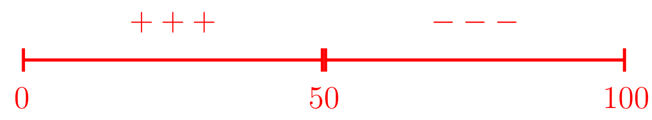
$$0 = \frac{dA}{dx}$$

$$0 = 450 - 9x$$

$$9x = 450$$

$$x = 50$$

Check for the values of A':



So we can see that there is an absolute maximum at $x = 50$. Evaluate y at $x = 50$

$$y = 150 - 1.5 * (50)$$
$$y = 75$$

$$\boxed{Area = 3 * 50 * 75 = 11,250 \text{ ft}^2}$$

7. Find $\frac{dy}{dx}$ given the following:

$$\sin\left(x^2 + y^3\right) = 5y + 8x$$

It is okay to leave your answer in terms of both x and y.

$$\frac{d}{dx}(\sin\left(x^2 + y^3\right)) = \frac{d}{dx}(5y + 8x)$$

$$\cos\left(x^2 + y^3\right) * (2x + 3y^2 * \frac{dy}{dx}) = 5 * \frac{dy}{dx} + 8$$

$$2x * \cos\left(x^2 + y^3\right) + 3y^2 * \frac{dy}{dx} * \cos\left(x^2 + y^3\right) = 5 * \frac{dy}{dx} + 8$$

$$(3y^2 \cos\left(x^2 + y^3\right) - 5)\frac{dy}{dx} = 8 - 2x\cos\left(x^2 + y^3\right)$$

$$\boxed{\frac{dy}{dx} = \frac{8 - 2x\cos(x^2 + y^3)}{3y^2\cos(x^2 + y^3) - 5}}$$

8. Evaluate the following derivatives:

(a) $\frac{d}{dx}\cos(x)$

(b) $\frac{d}{dx}\csc(x)$

(c) $\frac{d}{dx}\tan(x)$

(d) $\frac{d}{dx}\arcsin(x)$

(e) $\frac{d}{dx}\ln(x)$

(a) $\frac{d}{dx}\cos(x) = -\sin(x)$

(b) $\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$

(c) $\frac{d}{dx}\tan(x) = \sec^2(x)$

(d) $\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$

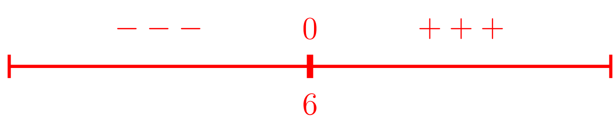
(e) $\frac{d}{dx}\ln(x) = \frac{1}{x}$

9. A function $f(x)$ has the first derivative $f'(x) = e^{0.5x}(10x - 60)$

(a) Upon which interval is $f(x)$ increasing?

(b) Upon which interval is the graph of $f(x)$ concave down?

(a) Based off of the graph below, we can say f is increasing on the interval $(6, \infty)$. So the answer is $\boxed{(6, \infty)}$.



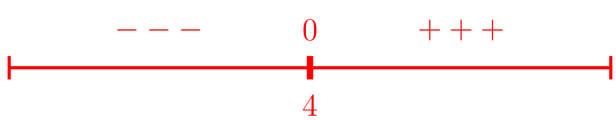
(b)

$$f''(x) = 0.5e^{0.5x} * (10x - 60) + e^{0.5x} * 10$$

$$f''(x) = e^{0.5x}(0.5(10x - 60) + 10)$$

$$f'(x) = e^{0.5x}(5x - 20)$$

Values of f"(x):



The function is concave down on the interval $\boxed{(-\infty, 4)}$

10. Evaluate each of the following limits:

(a)

$$\lim_{x\to\infty} \frac{2\ln(x)}{\sqrt[3]{x}}$$

(b)

$$\lim_{x\to 0} \frac{e^{10x}-1}{5x}$$

(c)

$$\lim_{x\to\infty} \frac{e^{10x}-1}{5x}$$

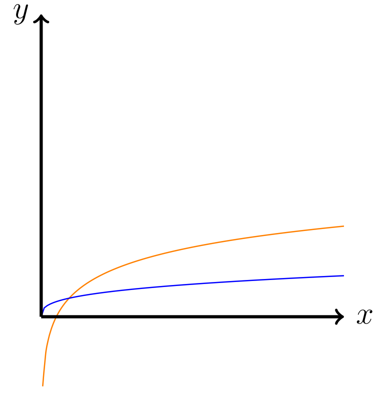
(a)

$$\lim_{x\to\infty} \frac{2\ln(x)}{\sqrt[3]{x}} = 2\lim_{x\to\infty} \frac{\ln(x)}{\sqrt[3]{x}}$$

Using l'Hopital's Rule:

$$2\lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{3}x^{-2/3}} = 2\lim_{x\to\infty} \frac{3}{x^{1/3}} = 2(0) = \boxed{0}$$

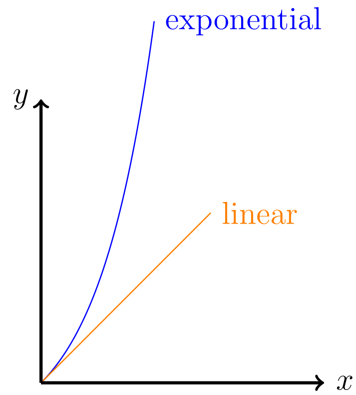
$$\left(\text{Blue} = \sqrt[3]{x},\ \text{orange} = 2\ln(x)\right)$$



(b) For this limit we must apply l'Hopital's Rule

$$\lim_{x\to 0} \frac{e^{10x}-1}{5x} = \lim_{x\to 0} \frac{10e^{10x}}{5} = \boxed{2}$$

(c) The numerator approaches infinity more rapidly than the denominator $\boxed{\lim_{x\to\infty} \to \infty}$



(Note about plots, these plots do not have the coefficients attached to x due to size constraints of the page. The general relation still holds and, in fact, is exacerbated by the coefficients)