Center for Academic Resources in Engineering (CARE)
Peer Exam Review Session

Math 285 – Intro Differential Equations

Midterm 2 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Tuesday, March 19th from 6:30-8pm in 2018 CIF Tutors: Vallabh and Pallab

Session 2: Thursday, March 21st from 5-6:30pm in 2018 CIF Tutors: Charlie and Eric

Can't make it to a session? Here's our schedule by course:

https://care.grainger.illinois.edu/tutoring/schedule-by-subject

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/846
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. The differential equation $mu'' + 4u' + 8u = 0$ describes a mass-spring system. For what values of m is the system underdamped?

A) $m < \frac{1}{2}$
B) $m < 2$
C) $m > \frac{1}{2}$
D) $m > 2$

The system is underdamped when its characteristic equation $mr^2 + 4r + 8 = 0$ has two complex roots. This happens when the discriminant $4^2 - 4 * m * 8 = 16 - 32m$ is negative.

So, $16 - 32m < 0$ thus $\boxed{m > \frac{1}{2}}$

2. For what forcing frequency ω is it possible for the solution to $18y'' + 2y = 81\cos(\omega t)$ to experience resonance?

A) $\omega = 3$
B) $\omega = 9$
C) $\omega = \frac{1}{3}$
D) $\omega = \frac{1}{9}$

When the system is in resonance the external frequency ω equals the resonance frequency, ω_0.

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{2}{18}} = \boxed{\frac{1}{3}}$$

3. What is the correct expression for the characteristic solution of the following ODE?

$$y'' - 8y' + 17y = 0$$

A) $y(t) = e^{-4t}[c_1 cos(t) + c_2 sin(t)]$
B) $y(t) = e^{4t}[c_1 cos(t) + c_2 sin(t)]$
C) $y(t) = e^{t}[c_1 cos(4t) + c_2 sin(4t)]$
D) $y(t) = e^{-t}[c_1 cos(-4t) + c_2 sin(-4t)]$

The characteristic equation takes the form of:

$$r^2 - 8r + 17 = 0$$

Solving for r with the quadratic formula gives the following roots:

$$r = \frac{8 \pm \sqrt{64 - (4)(1)(17)}}{2} = 4 \pm 1i$$

Complex roots require the solution to take the form $y(t) = e^{at}[c_1 cos(bt) + c_2 sin(bt)]$. Therefore, the answer is B.

4. Consider the following statements

(i) If Y_1 is a particular solution to the DE

$$y'' + p(t)y' + q(t)y = g(t)$$

then $Y_2 = cY_1$ is also a solution to the differential equation, where c is a constant

(ii) If y_1 and y_2 are solutions to

$$ay'' + by' + cy = g(t)$$

then y_1 - y_2 is a solution to the homogeneous differential equation $ay'' + by' + cy = 0$ where a, b, and c are constants.

(iii) The solution to

$$ay'' + by' + cy = 0$$

with initial conditions $y(0) = A$ and $y'(0) = B$, is unique on $t \in (-\infty, \infty)$ where a, b, and c are constants.

Which of the above statements are always true?

A) (i) and (ii)
B) (i) and (iii)
C) (ii) and (iii)
D) None

(i) This statement is False:

$$Y_2'' + p(t)Y_2' + q(t)Y_2 = g(t)$$
$$c(Y_1'' + p(t)Y_1' + q(t)Y_1) = cg(t)$$
$$cg(t) \neq g(t) \text{ if c } \neq 0 \text{ not ALWAYS true}$$

(ii) This statement is True:

$$a(y_1 - y_2)'' + b(y_1 - y_2)' + c(y_1 - y_2)$$
$$(ay_1'' + by_1' + cy_1) - (ay_2'' + by_2' + cy_2) = g(t) - g(t) = 0$$

(iii) This statement is True:

The existence and uniqueness theorem states that constant functions are continuous. The initial conditions also exist on the interval.

The answer is therefore **(C)**.

5. Consider the following initial value problem $(t-5)y'' + \csc(t)y' + y = e^t$, $y'(4) = 1$ and $y(4) = \pi$. What is the largest interval on which the initial value problem is guaranteed to exist?

   A) $(0,\ 5)$

   B) $(0,\ 2\pi)$

   C) $(\pi,\ 5)$

   D) $(\pi,\ 2\pi)$

   Rewrite the equation as

$$y'' + \frac{\csc(t)}{(t-5)}y' + \frac{1}{(t-5)}y = \frac{e^t}{(t-5)}$$

   The coefficients exist everywhere except $t = 5$ and $t = 0,\ \pi,\ 2\pi...$
   We need to find the interval that contains the point 4 and none of the points that don't exist, which is **(C)**.

6. Identify the form of the particular solution for $y'' - 16y = (t-3)e^{-4t} + (4t+3)\sin(2t)$

   A) $(At+B)te^{-4t} + (Ct+D)t\sin(2t) + (Et+F)t\cos(2t)$
   B) $(At+B)e^{-4t} + (Ct+D)\sin(2t) + (Et+F)\cos(2t)$
   C) $(At+B)e^{-4t} + (Ct+D)t\sin(2t) + (Et+F)t\cos(2t)$
   D) $(At+B)te^{-4t} + (Ct+D)\sin(2t) + (Et+F)\cos(2t)$

   The particular solution, y_p, should be of the same form as the right side of the equation. We might need to multiply by a power of t to ensure linear independence from the characteristic solution. This means that the particular solution expression should not have any of same terms in the characteristic solution.

   $$y_c = C_1e^{4t} + C_2e^{-4t}$$

   The term $(t-3)e^{-4t}$ corresponds to the following particular solution term (we multiply by t to avoid these duplicate terms):

   $$(At+B)te^{-4t}$$

   While the term $(4t+3)\sin(2t)$ corresponds to the following particular solution term:

   $$(Ct+D)\sin(2t) + (Et+F)\cos(2t)$$

   Combine these two solutions to get the form of the solution, which is **(D)**.

7. Find the general solution for the differential equation $y'' + 4y' + 4y = 0$

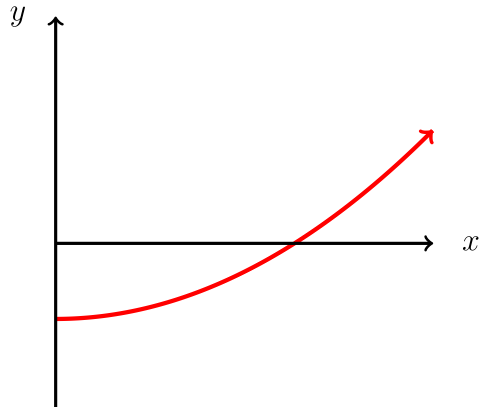
   First, find the solution to the homogeneous equation:

   $$r^2 + 4r + 4 = 0$$

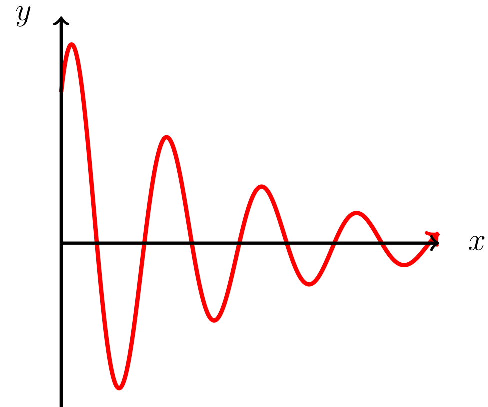
   This has a repeated root of $r = -2$.
   Then, we multiply one of the solutions by t to obtain:

   $$\boxed{y = C_1e^{-2t} + C_2te^{-2t}}$$

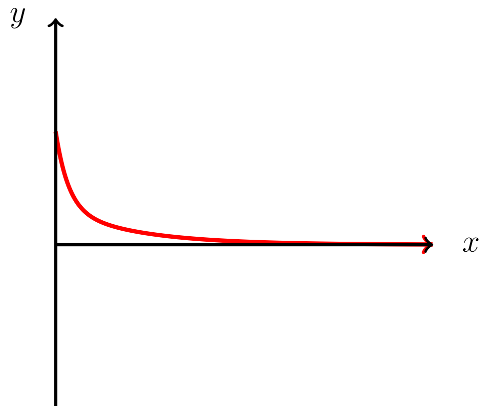
8. Which of the following plots represents a solution to the ODE $y'' + 7y' + 6y = 0$?



(Plot A)



(Plot B)



(Plot C)

A) Plot A
B) Plot B
C) Plot C
D) None of the Above

The characteristic equation

$$r^2 + 7r + 6 = 0$$

has roots $r = -1$ and $r = -6$
The solution is

$$y = c_1 e^{-x} + c_2 e^{-6x}$$

The solution tends to 0 due to the exponential decay terms, which gives the answer **(C)**.

9. Use the method of undetermined coefficients to find the general solution to the following ODE:

$$y'' - 4y = e^{-2t}$$

The characteristic equation

$$r^2 - 4 = 0$$

has roots $r = -2$ and $r = 2$. The characteristic equation is

$$y_c = c_1 e^{-2t} + c_2 e^{2t}$$

The particular solution is

$$y_p = Ate^{-2t}$$
$$y_p' = -2Ate^{-2t} + Ae^{-2t}$$
$$y_p'' = 4Ate^{-2t} - 4Ae^{-2t}$$

Notice that you need to multiply by t to maintain linear independence. Now plug y_p into the original ODE:

$$4Ate^{-2t} - 4Ae^{-2t} - 4Ate^{-2t} = e^{-2t}$$
$$-4Ae^{-2t} = e^{-2t}$$
$$A = \frac{-1}{4}$$
$$y_p = \frac{-1}{4} te^{-2t}$$

To find the general solution, add y_c and y_p

$$y(t) = y_c + y_p$$

$$\boxed{y(t) = c_1 e^{-2t} + c_2 e^{2t} - \tfrac{1}{4} te^{-2t}}$$

10. Find the solution $y(t)$ for the following ODE with a Laplace transformation. Assume all initial conditions are zero.

$$3y' + 4y = t$$

The ODE can be rewritten as

$$3(sY(s) - y(0)) + 4Y(s) = \frac{1!}{s^2}$$

By neglecting the $y(0)$ term and combining like terms, the function can be written as

$$Y(s) = \frac{1}{s^2(3s+4)}$$

This is solved using partial fractions decomposition where

$$\frac{1}{s^2(3s+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{3s+4}; \qquad 1 = As(3s+4) + B(3s+4) + Cs^2$$

When $s = 0$, $1 = 4B$ so $B = \frac{1}{4}$.
When $s = -\frac{4}{3}$, $1 = (\frac{-4}{3})^2 C$ so $C = \frac{9}{16}$
When $s = 1$

$$1 = A(7) + \frac{1}{4}(7) + \frac{9}{16}$$

$$1 = 7A + \frac{7}{4} + \frac{9}{16}$$

$$1 = 7A + \frac{28}{16} + \frac{9}{16}$$

$$1 = 7A + \frac{37}{16}$$

$$\frac{16}{16} - \frac{37}{16} = 7A$$

$$\frac{-21}{16} = 7A$$

$$A = \frac{-3}{16}$$

$$Y(s) = \frac{-3}{16s} + \frac{1}{4s^2} + \frac{9}{16(3s+4)}$$

This can be simplified by isolating s in the last term

$$Y(s) = \frac{-3}{16s} + \frac{1}{4s^2} + (\frac{1}{16})\frac{3}{(s+\frac{4}{3})}$$

Using the table of Laplace transforms gives

$$\boxed{y(t) = \frac{-3}{16} + \frac{1}{4}t + \frac{3}{16}e^{\frac{-4}{3}t}}$$

11. Find the general solution to the following ODE using variation of parameters.

$$y'' + 2y' + y = e^{-t}$$

The complementary solution takes the form

$$y_c = C_1 e^{-t} + C_2 t e^{-t}$$

Taking $y_1 = e^{-t}$ and $y_2 = te^{-t}$, the Wronksian W is equal to e^{-2t}. These values should be substituted into the variation of parameters formula with $g(t) = e^{-t}$

$$Y_p(t) = -y_1 \int \frac{y_2 g(t)}{W(y_1, y_2)} dt + y_2 \int \frac{y_1 g(t)}{W(y_1, y_2)} dt$$

This simplifies to

$$Y_p(t) = -e^{-t} \int t dt + te^{-t} \int 1 dt = \frac{1}{2} t^2 e^{-t}$$

The general solution is therefore

$$\boxed{y(t) = y_c + Y_p = C_1 e^{-t} + C_2 t e^{-t} + \frac{1}{2} t^2 e^{-t}}$$

12. Calculate the Wronskian of $y_1(t) = e^{-3t^2-3t+6}$ and $y_2(t) = e^{-3t^2-3t-3}$ and explain if your result shows linear independence, linear dependance, or neither:

The Wronskian result is the following:

$$W[y_1, y_2](t) = y_1 y_2' - y_1' y_2 = e^9 y_2 y_2' - e^9 y_2 y_2' = \boxed{0}$$

Notice that $y_1(t) = e^9 y_2(t)$. If $y_1(t)$ is a scalar multiple of $y_2(t)$, then $y_1(t)$ and $y_2(t)$ are linear dependent. IMPORTANT: A Wronskian of 0 does not automatically prove linear dependence. You must show that the functions are scalar multiples if each other to prove linear dependance. The Wronskian only tests for linear independence.