MATH 257 Mid-semester CARE Review

Please join the queue!

https://queue.illinois.edu/q/queue/955

In-Person Resources

CARE Drop-in tutoring:
7 days a week on the 4th floor of Grainger Library!
Sunday - Thursday 12pm-10pm
Friday & Saturday 12-6pm

Course Office hours:
TAs: Monday - Thursday 5-7pm in English Building 108
Instructors: Chuang MW 4-5PM in CAB 233
Leditzky MW 2:30-3:30PM in CAB 39
Luecke Tu 1:30-3:30PM in Altgeld 105

Subject	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
Math 257	3pm-9pm	2pm-4pm 8pm-10pm	2pm-6pm 8pm-10pm	2pm-5pm 6pm-10pm	2pm-9pm	1pm-6pm	12pm-6pm

Structure

First hour: Midterm 1 Content

30 mins: interactive big group presentation

30 mins: small group problem solving

Second hour: Midterm 2 Content

30 mins: interactive big group presentation

30 mins: small group problem solving

Last hour: Midterm 2+ Content (midterm 3 content so far) – go to the midterm 3 review session for worksheet/problems!

How has the class been so far?

Which statement best describes you?

1. I don’t understand the concepts at all.
2. I’m not sure how to approach conceptual questions/what concepts are relevant.
3. I’m struggling to do all of the questions in time.

Midterm 1 Topics

- Linear systems
  - Solving systems with matrices
- Reduced row echelon form
  - Pivot columns: basic and free variables
  - Row operations
- Vectors and spans
- Matrix operations
  - Addition, subtraction, scalar multiplication, linear combinations
  - Transposition
- Matrix multiplication
  - Properties of matrix multiplication
- Matrix inverses
  - What matrices are invertible?
  - Elementary matrices

Linear Systems

$$a_1x_1 + \ldots + a_nx_n = b$$

and matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Linear systems must have either:

1. One unique solution
2. Infinite solutions
3. No solutions

Equivalent linear systems have the same set of solutions.

You can represent a linear system with matrices…

linear system	coefficient matrix	augmented matrix

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Column vector

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Row vector

We often define a matrix in terms of its columns or its rows:

$\mathbf{a}_n$ are all column vectors

$$A := \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \qquad or \qquad A := \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \vdots \\ \mathbf{R}_m \end{bmatrix}$$

$\mathbf{R}_m$ are all row vectors

Echelon Forms

Pivot

$$\begin{bmatrix} 3 & 1 & 2 & 0 & 5 \\ 0 & 2 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Free

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \end{bmatrix}$$

Row Echelon Form (REF):

1. All nonzero rows above rows of all zeros
2. Leading entry (leftmost nonzero number) is strictly to the right of the leading entry of the row above

Reduced Row Echelon Form (RREF):

3. Leading entries of nonzero rows are all 1
4. Each leading entry is the only nonzero entry in the column

Gaussian Elimination (for a general solution)

General Process

1. Write down the augmented matrix.
2. Find the RREF of the matrix.
3. Write down linear equations based on the RREF.
4. Express pivot variables in terms of free variables (unless there are no free variables).
5. Solve only if there are no **free variables** and the matrix is **consistent**. (This means the solution is unique!)

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \leftarrow$$

$0 = 0$ ✓

Dependent

AKA consistent

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{array}\right] \leftarrow$$

$0 = 2$ ✗

Inconsistent

Inconsistent systems have no solutions.

Elementary Row Operations

Elementary operations do not change the solution set of a system.

There are three kinds:

1. Replacement ($R_1 \rightarrow R_1 + a*R_2$)
2. Scaling ($R_1 \rightarrow a*R_1$)
3. Interchange ($R_1 \rightarrow R_2$)

All elementary operations are reversible. Two matrices are **row equivalent** if elementary operations can turn one into the other.

Matrix Operations

a) **The sum of** $A + B$ is

$$\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

b) **The product** cA for a scalar c is

$$\begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}$$

Addition: only defined for matrices with the **same dimensions**

Subtraction: the same as addition

Scalar multiplication: every entry is multiplied by the **scalar**

- Scalar = any real number

Linear combinations: any mixture of scalar multiplication and addition/subtraction of matrices

- **span(a,b)** is a set of ALL the possible linear combinations of **a** and **b**

8. *Consider the set*

$$W := \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3 \ : \ a_1 + a_2 - a_3 = 0 \right\}$$

Can you write W as a span of two vectors?

Intuitive solution: can you find two independent vectors that satisfy the given relationship?

Rigorous solution: write the equation on the left as an augmented matrix. Find the general solution in parametric form.

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \end{array}\right] \longrightarrow \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -a_2 + a_3 \\ a_2 \\ a_3 \end{bmatrix} = a_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

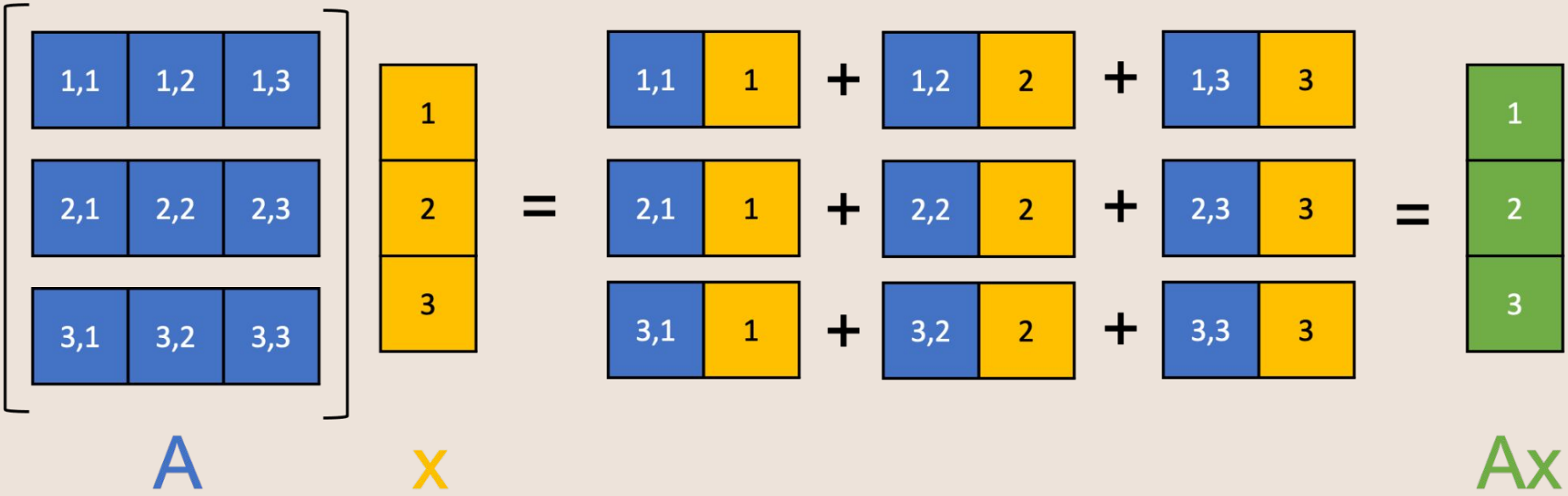
Matrix Operations (cont.)

Transpose: switch rows and columns

2	4	-1
-10	5	11
18	-7	6

2	-10	18
4	5	-7
-1	11	6

Matrix-vector multiplication: A**x** = $x_1\mathbf{a_1} + x_2\mathbf{a_2} + \ldots + x_n\mathbf{a_n}$ which means you multiply the **entries** of the vector with the **columns** of the matrix



Matrix-vector multiplication

- The number of entries in **x** must match the number of columns in A

Matrix multiplication

Only defined for two matrices A and B if

- A has the dimensions *m* x *n* and B has the dimensions *n* x *p*
- A^k (exponent) is only defined for a **square** matrix

Each entry of AB is a linear combination of a **row of A** with a **column of B.**

$$AB = \begin{bmatrix} \mathbf{R}_1\mathbf{C}_1 & \dots & \mathbf{R}_1\mathbf{C}_p \\ \mathbf{R}_2\mathbf{C}_1 & \dots & \mathbf{R}_2\mathbf{C}_p \\ \mathbf{R}_m\mathbf{C}_1 & \dots & \mathbf{R}_m\mathbf{C}_p \end{bmatrix} \quad \text{and} \quad (AB)_{ij} = \mathbf{R}_i\mathbf{C}_j = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

Properties of Matrix Multiplication

(a) $A(BC) = (AB)C$ *(associative law of multiplication)*

(b) $A(B + C) = AB + AC$, $(B + C)A = BA + CA$ *(distributive laws)*

(c) $r(AB) = (rA)B = A(rB)$ *for every scalar* r,

(d) $A(rB + sC) = rAB + sAC$ *for every scalars* r, s *(linearity of matrix multiplication)*

(e) $I_m A = A = AI_n$ *(identity for matrix multiplication)*

Transpose Theorem: $(AB)^T = B^T A^T$

Matrix multiplication is NOT COMMUTATIVE: AB ≠ BA

11. *Let* $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ *be a* 3×3*-matrix such that the columns of* A *sum up to the zero vector (i.e.* $\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 = \mathbf{0}$*), and let* $B = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$*. What can you say about the product matrix* AB*?*

Always write out the product if you don't see a pattern at first!

$$AB = \begin{bmatrix} 0\mathbf{a}_1 + 0\mathbf{a}_2 + 0\mathbf{a}_3 & \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 & 2\mathbf{a}_1 + 2\mathbf{a}_2 + 2\mathbf{a}_3 \end{bmatrix}$$

Try to relate back to the information given in the question → AB is the zero matrix

Elementary Matrices

Identity Matrices

$1 \times 1 \quad [1]$

$2 \times 2 \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$3 \times 3 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

etc.

Any matrix that can be form from the identity matrix with **one** elementary row operation.

Ex.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{R_2 \to 3R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{R_2 \leftrightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Elementary Matrices (cont.)

Multiplying an elementary matrix (E) with another matrix (A) is the same as performing the elementary row operation on A.

This means you can represent putting a matrix in RREF as a sequence of matrix multiplications:

$E_n \ldots E_2 E_1 A = B$ where A is the original matrix and B is the RREF form

6. *Let* $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. *Using what you know about elementary matrices, determine* A^n *for any natural number* n.

What do you notice about A? What elementary row operation does A do?

R1 → R1 + R2

Each time you multiply A with itself, it will add one to the top right corner:

$$A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

Matrix Inverses

Determinants:

$$\frac{1}{ad - bc}$$

For the matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Definition of an inverse:

$$AC = I_n$$

Requirements for a matrix to be invertible:

1. It has to be square
2. The determinant of the matrix cannot be 0 or
3. The RREF of A is the identity matrix or
4. A has as many pivots as columns/rows

Statements 2, 3, and 4 mean the same thing.

Calculating an Inverse

For 2x2:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Elementary Matrix strategy:

$$A^{-1} = E_m E_{m-1} \dots E_1 = E_m E_{m-1} \dots E_1 I_n.$$

OR: set up an augmented matrix with the identity and reduce to RREF

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{array}\right]$$

Works for any square matrices of any size

Properties of Matrix Inverses

(a) A^{-1} *is invertible and* $\left(A^{-1}\right)^{-1} = A$ *(i.e. A is the inverse of A^{-1}).*

(b) AB *is invertible and* $(AB)^{-1} = B^{-1}A^{-1}$.

(c) A^T *is invertible and* $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

Inverses are unique! Every invertible matrix only has one inverse.

Multiplying by a matrix inverse is the closest we get to dividing matrices.

Theorem 14. *Let A be an invertible $n \times n$ matrix. Then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.*

Is the sum of two invertible matrices always an invertible matrix? Justify your answer.

Always look for counterexamples!

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Both have non-zero determinants and pivots in all columns which means they are invertible, but the sum is the zero matrix, which is not invertible! Simple matrices like the identity matrix and zero matrix are often good counterexamples to try.

Small Group Time!

- Worksheets are organized by topic! Pick one or multiple to work through!

Midterm 2 Topic Summary

- LU Decomposition
  - Lower/Upper Triangular Matrix
  - LU for Linear Systems
  - Permutation Matrix
- Vectors and Spans
  - Inner Product
  - Orthogonality
  - Linear Independence
- Subspaces
  - Column Space
  - Null Space
- Basis and Dimension
  - Fundamental Subspaces
  - Orthonormal bases
  - Orthogonal/normal Complements
- Graph and Adjacency Matrices
- Coordinates
  - Coordinate Matrices

Upper/Lower Triangular Matrices

Upper Triangular:

$$\begin{bmatrix} \star & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & * \end{bmatrix}$$

Finding this is like doing REF with only row replacement

Lower Triangular:

$$\begin{bmatrix} \star & 0 & 0 & 0 & 0 \\ \star & \star & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 \\ \star & \star & \star & \ddots & \vdots \\ \star & \star & \star & \star & \star \end{bmatrix}$$

Keep track of your row operations to find L

LU Decomposition:

$$A = LU$$

- Not all matrices have LU decompositions
- LU decompositions are not unique (unlike inverses)

LU Decomposition Example

Determine the LU-decomposition of $\begin{bmatrix} 1 & 2 & 2 \\ 4 & 4 & 4 \\ 4 & 4 & 8 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 2 \\ 4 & 4 & 4 \\ 4 & 4 & 8 \end{bmatrix} \underset{R_2 \to R_2 \boxed{-4} R_1,\, R_3 \to R_3 \boxed{-4} R_1}{\overset{\text{1) Col 1 Row 2} \quad \text{2) Col 1 Row 3}}{\rightsquigarrow}} \begin{bmatrix} 1 & 2 & 2 \\ 0 & -4 & -4 \\ 0 & -4 & 0 \end{bmatrix} \underset{R_3 \to R_3 \boxed{-1} R_2}{\overset{\text{3) Col 3 Row 3}}{\rightsquigarrow}} \begin{bmatrix} 1 & 2 & 2 \\ 0 & -4 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

$$, L := \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix}$$

$$U := \begin{bmatrix} 1 & 2 & 2 \\ 0 & -4 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

LU for Linear Systems

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}}_{L} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}}_{\mathbf{b}}$$

$$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix}}_{\mathbf{c}}.$$

Use **LU decomposition** to solve a linear system if:

1. A is nxn matrix
2. A = LU
3. b ∈ R^n

Step-by-step Algorithm

1. Find L and U
2. Solve for c using Lc = b
3. Solve for x using Ux = c

$$A\mathbf{x} = \mathbf{b}$$

$$L\mathbf{c} = \mathbf{b} \longrightarrow U\mathbf{x} = \mathbf{c}$$

$$A\mathbf{x} = (LU)\mathbf{x} = L(U\mathbf{x}) = L\mathbf{c} = \mathbf{b}$$

Permutation Matrices: for matrices that don't have an LU decomposition

Theorem 21. *Let A be $n \times n$ matrix. Then there is a permutation matrix P such that PA has an LU-decomposition.*

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix} \underset{R_1 \leftrightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = PA$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Step-by-step:

- Use the interchange operation done on **A** to an equivalent size identity matrix, this will be your **P** matrix
- Solve the for the **LU** decomposition of **PA**

When we apply the $\mathbf{P}^{-1}$ to **LU** (on the right), we'll be able to get the original value of **A**

$$\boxed{PA} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ .5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=:L} \underbrace{\begin{bmatrix} 2 & 1 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=:U}$$

Inner Product, Norm, and Distance

If $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$, then $\mathbf{v} \cdot \mathbf{w}$ is

$$v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

The **inner product** of $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ is

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w}$$

AKA the dot product

It is a scalar!

Definition. Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

The **norm** (or **length**) of $\mathbf{v}$ is

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + \cdots + v_n^2}.$$

The **distance** between $\mathbf{v}$ and $\mathbf{w}$ is

$$\operatorname{dist}(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|.$$

The norm is also a scalar!

Properties of the Inner Product: similar to scalars

Theorem 22. *Let* $\mathbf{u}, \mathbf{v}$ *and* $\mathbf{w}$ *be vectors in* $\mathbb{R}^n$*, and let* c *be any scalar. Then*

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ Commutative!

(b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$ Distributive!

(c) $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$ Associative!

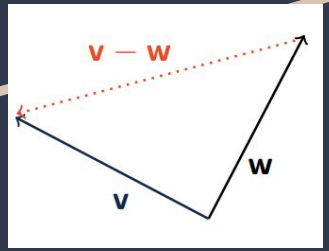
(d) $\mathbf{u} \cdot \mathbf{u} \geq \mathbf{0}$, *and* $\mathbf{u} \cdot \mathbf{u} = \mathbf{0}$ *if and only if* $\mathbf{u} = \mathbf{0}$.

Orthogonality

(fancy word for perpendicular)

Vectors are orthogonal if their **dot product is zero**.

Why? The dot product of two non-zero vectors can only be zero if the angle between them is 90.



Orthonormality

A **unit vector** in $\mathbb{R}^n$ is vector of length 1.

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

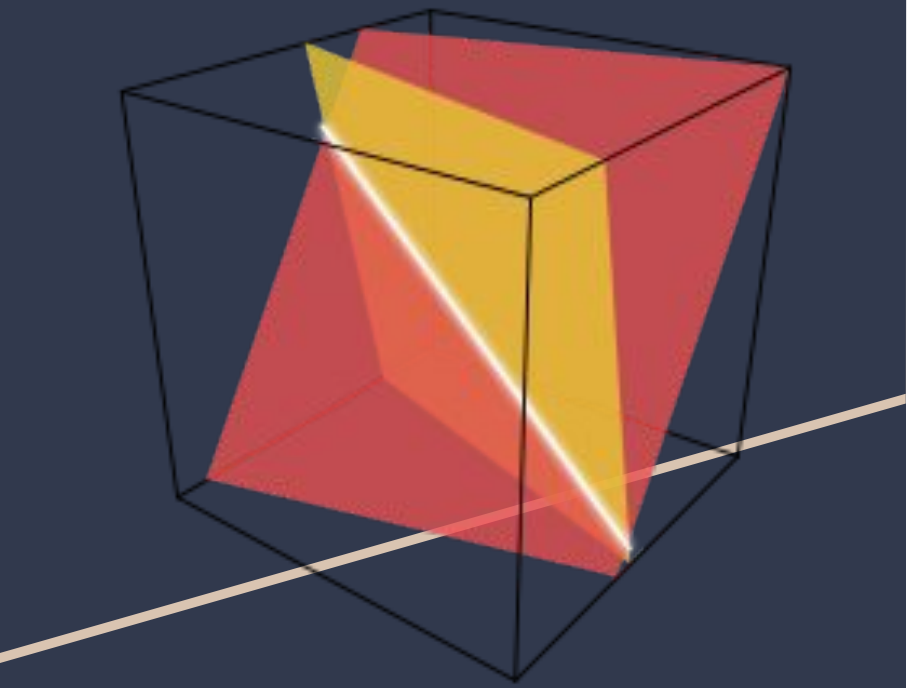
Orthonormal sets are all orthogonal to each other and unit vectors.

Ex.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{u_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{u_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Subspaces



W is a **subspace** of **V,** if:

- W contains the 0 vector
- Adding any 2 vectors in W together gives a vector also in W
- Multiplying any vector in W by any scalar gives a vector also in W

Theorem 24. *Let* $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m} \in \mathbb{R}^n$. *Then Span* $(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m})$ *is a subspace of* $\mathbb{R}^n$.

Vector Spaces 'V': a specific type of subspace

$u, v, w \in V$ and for all scalars $c, d \in \mathbb{R}$:

- $\mathbf{u} + \mathbf{v}$ is in V. (V is "closed under addition".)
- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
- There is a vector (called the zero vector) $\mathbf{0}_V$ in V such that $\mathbf{u} + \mathbf{0}_V = \mathbf{u}$.
- For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V satisfying $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}_V$.
- $c\mathbf{u}$ is in V. (V is "closed under scalar multiplication".)
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
- $(cd)\mathbf{u} = c(d\mathbf{u})$.
- $1\mathbf{u} = \mathbf{u}$.

Counterexamples

(a) $W_1 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a - 2b = c, 4a + 2c = 0 \right\} \subseteq \mathbb{R}^3.$

(b) $W_2 = \left\{ \begin{bmatrix} a - b \\ c \\ a + c \\ a - 2b - c \end{bmatrix} : a, b, c \in \mathbb{R} \right\} \subseteq \mathbb{R}^4.$

(c) $W_3 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a \cdot b \geq 0 \right\} \subseteq \mathbb{R}^2.$

(d) $W_4 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : b \geq 0 \right\} \subseteq \mathbb{R}^2.$

(e) $W_5 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a - 2b = 1, 4a + 2c = 0 \right\} \subseteq \mathbb{R}^3.$

W_3 : not closed under addition

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$(-1)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

W_4 : not closed under scalar multiplication

Which are subspaces of the given vector spaces?

(a) $W_1 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a - 2b = c, 4a + 2c = 0 \right\} \subseteq \mathbb{R}^3.$

(b) $W_2 = \left\{ \begin{bmatrix} a - b \\ c \\ a + c \\ a - 2b - c \end{bmatrix} : a, b, c \in \mathbb{R} \right\} \subseteq \mathbb{R}^4.$

(c) $W_3 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a \cdot b \geq 0 \right\} \subseteq \mathbb{R}^2.$

(d) $W_4 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : b \geq 0 \right\} \subseteq \mathbb{R}^2.$

(e) $W_5 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a - 2b = 1, 4a + 2c = 0 \right\} \subseteq \mathbb{R}^3.$

W_1 : relate to a Nullspace

$$\begin{bmatrix} 1 & -2 & -1 \\ 4 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & -2 & -1 \end{bmatrix}$$

W_2 : relate to a Column space

W_5: does not contain the zero vector

Relating existing known subspaces to prove it is a subspace

Column Spaces

Definition. The **column space**, written as $\mathrm{Col}(A)$, of an $m \times n$ matrix A is the set of all linear combinations of the columns of A. If $A = \begin{bmatrix} \mathbf{a_1} & \mathbf{a_2} & \cdots & \mathbf{a_n} \end{bmatrix}$, then $\mathrm{Col}(A) = \mathrm{span}\,(\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n})$.

$$A = \begin{bmatrix} 1 & -10 & -24 & -42 \\ 1 & -8 & -18 & -32 \\ -2 & 20 & 51 & 87 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -10 & -24 & -42 \\ 1 & -8 & -18 & -32 \\ -2 & 20 & 51 & 87 \end{bmatrix} \xrightarrow[R_3 + 2R_1 \to R_3]{R_2 - R_1 \to R_2} \begin{bmatrix} \underline{1} & -10 & -24 & -42 \\ 0 & \underline{2} & 6 & 10 \\ 0 & 0 & \underline{3} & 3 \end{bmatrix}$$

$$\mathrm{Col}(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -10 \\ -8 \\ 20 \end{bmatrix}, \begin{bmatrix} -24 \\ -18 \\ 51 \end{bmatrix} \right\}$$

How to solve for Col(A):

1. Put matrix A into REF
2. Find all the pivots of A
3. Map the pivots to the columns of your original matrix, A

Null Spaces

Definition. The **nullspace** of an $m \times n$ matrix A, written as $\mathrm{Nul}(A)$, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$; that is, $\mathrm{Nul}(A) = \{\mathbf{v} \in \mathbb{R}^n \ : \ A\mathbf{v} = \mathbf{0}\}$.

How to solve for Nul(A):

1. Set matrix A into **Augmented Matrix** with zeros on the right (**Ax = 0**)
2. Get A into **RREF**
3. Solve for **x**

Nul(A) = span($\mathbf{x}_1$, $\mathbf{x}_2$,...)

Construct a matrix A such that

$$\mathrm{Nul}(A) = \mathrm{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}\right)$$

Step 1. What are the dimensions of A?

A has 3 columns, since the vectors have 3 entries

Step 2. How many pivots/free variables does A have?

2 free variables + 1 pivot variable

Step 3.

$$\begin{bmatrix}a & b & c\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = 0 \qquad \begin{bmatrix}a & b & c\end{bmatrix}\begin{bmatrix}1\\-1\\1\end{bmatrix} = 0 \longrightarrow \begin{bmatrix}1 & 0 & -1\end{bmatrix}$$

Linear Independence

Definition. Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are said to be **linearly independent** if the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution (namely, $x_1 = x_2 = \cdots = x_p = 0$).
We say the vectors are **linearly dependent** if they are not linearly independent.

Theorem 30. *Let A be an $m \times n$ matrix. The following are equivalent:*

- *The columns of A are linearly independent.*
- *$A\mathbf{x} = \mathbf{0}$ has only the solution $\mathbf{x} = \mathbf{0}$.*
- *A has n pivots.*
- *there are no free variables for $A\mathbf{x} = \mathbf{0}$.*

Basis and Dimension

Definition. Let V be a vector space. A sequence of vectors $(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ in V is a **basis** of V if

- $V = \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_p)$, and
- $(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ are linearly independent.

The number of vectors in a basis of V is the **dimension** of V.

Theorem 33. *A basis is a minimal spanning set of V; that is the elements of the basis span V but you cannot delete any of these elements and still get all of V.*

Find a basis for the following subspaces:

(i)

$$W_3 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2} : a = d = 0 \right\}.$$

(ii)

$$W_4 = \left\{ p(t) \in \mathbb{P}_2 : \frac{d}{dt} p(t) = 0 \right\}.$$

(Recall that $\mathbb{P}_2$ is the vector space of polynomials of degree at most 2).

(i) The dimension is only 2 since a and d are fixed. Pick an easy basis!

$$\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

(ii) First write a general expression for P_2 and its derivative to find:

$$W_2 = \{a_0 + a_1 t + a_2 t^2 : a_1 = a_2 = 0\} \longrightarrow$$ Only constants are allowed: {1}

Basis and Dim of four subspaces:

Rank [r] : Number of pivots matrix has

Let A be an $m \times n$ matrix with rank r

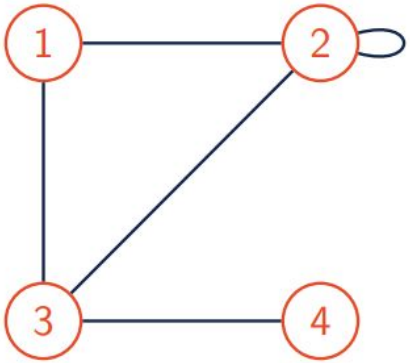
- dim Nul(A) = n - r
- dim Col(A) = r
- dim Nul(A^T) = m - r
- dim Col(A^T) = r

Graphs and Adjacency Matrices

A **graph** is a set of nodes (or: vertices) that are connected through edges.

Definition. Let $\mathcal{G}$ be a graph with n nodes. The **adjacency matrix** of $\mathcal{G}$ is the $n \times n$-matrix $A = (a_{ij})$ such that

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge between node } i \text{ and node } j \\ 0 & \text{otherwise .} \end{cases}$$

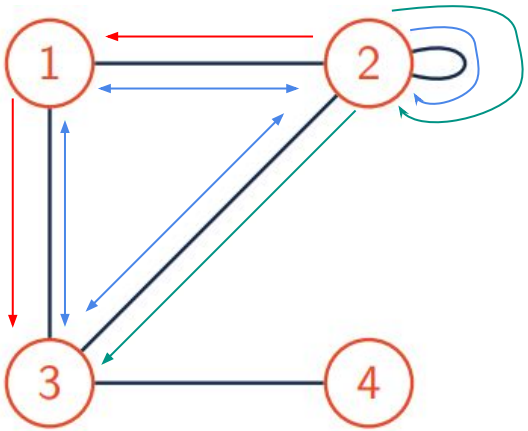


$$\begin{array}{cc} & \begin{array}{cccc} \text{N1} & \text{N2} & \text{N3} & \text{N4} \end{array} \\ & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Node 1: Connected to N2 \& N3} \\ \text{Node 2: Connected to N1, N2, \& N3} \\ \text{Node 3: Connected to N1, N2 \& N4} \\ \text{Node 4: Connected to N3} \end{array}$$

Walks and Paths

Definition. A **walk of length** k on a graph of is a sequence of $k+1$ vertices and k edges between two nodes (including the start and end) that may repeat.A **path** is walk in which all vertices are distinct.

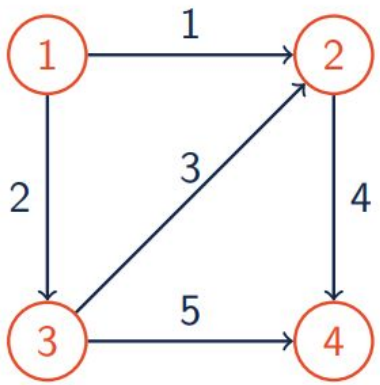
Example. Count the number of walks of length 2 from node 2 to node 3 and the number of walks of length 3 from node 3 back to node 3:



- Node 2 to Node 3: 2 walks of length 2
- Node 3 to Node 3: 3 walks of length 3

Directed Graphs

Definition. A **directed graph** is a set of vertices connected by edges, where the edges have a direction associated with them.



$$\begin{array}{cccc} \text{N1} & \text{N2} & \text{N3} & \text{N4} \\ \end{array}$$
$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Node 1: Nothing pointing to N1

Node 2: N1 and N3 pointing to N2

Node 3: N1 points to N3

Node 4: N2 and N3 pointing to N4

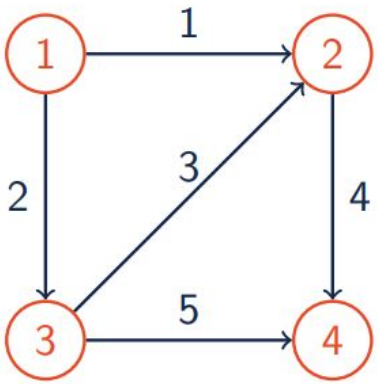
Definition. Let G be a directed graph with m edges and n nodes. The **adjacency matrix** of G is the $n \times n$ matrix $A = (a_{i,j})_{i,j}$ with

$$a_{i,j} = \begin{cases} 1, & \text{if there is a directed edge from node } j \text{ to node } i \\ 0, & \text{otherwise} \end{cases}$$

Edge-Node Incidence

Definition. Let G be a directed graph with m edges and n nodes. The **edge-node incidence matrix** of G is the $m \times n$ matrix $A = (a_{i,j})_{i,j}$ with

$$a_{i,j} = \begin{cases} -1, & \text{if edge } i \text{ leaves node } j \\ +1, & \text{if edge } i \text{ enters node } j \\ 0, & \text{otherwise} \end{cases}$$



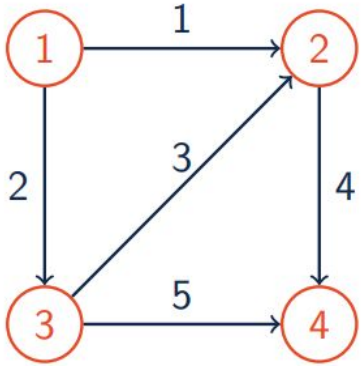
$$\begin{array}{cccc} \text{N1} & \text{N2} & \text{N3} & \text{N4} \\ \end{array}$$
$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Edge 1: Leaves N1; Enters N2
Edge 2: Leaves N1; Enters N3
Edge 3: Leaves N3; Enters N2
Edge 4: Leaves N2; Enters N4
Edge 5: Leaves N3; Enters N4

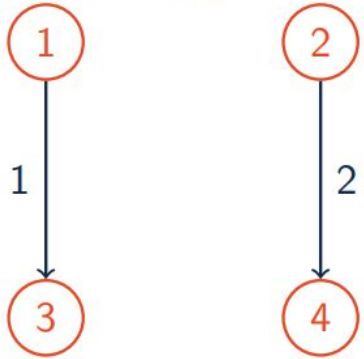
‘Connectedness’

Definition. A **connected component** of an undirected graph is a part in which any two vertices are connected to each other by paths, and which is connected to no additional vertices in the rest of the graph. The connected components of a directed graph are those of its underlying undirected graph. A graph is **connected** if only has one connected component.

A graph with one connected component:



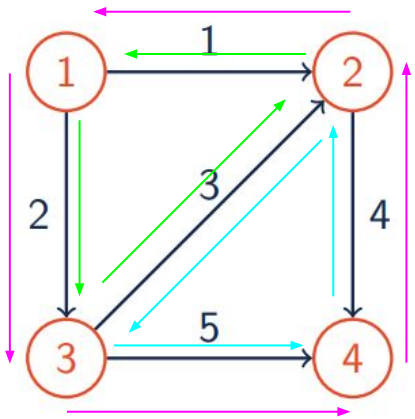
A graph with two connected components:



Theorem 40. *Let $\mathcal{G}$ be a directed graph and let A be its edge-node incidence matrix. Then* $\dim \mathrm{Nul}(A)$ *is equal to the number of connected components of $\mathcal{G}$.*

Cycles

Definition. A **cycle** in an undirected graph is a path in which all edges are distinct and the only repeated vertices are the first and last vertices. By cycles of a directed graph we mean those of its underlying undirected graph.



Cycle 1 Cycle 2 Cycle 3

$$\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Theorem 41. *Let $\mathcal{G}$ be a directed graph and let A be its edge-node incidence matrix. Then the cycle space of $\mathcal{G}$ is equal to* $\mathrm{Nul}(A^T)$.

Orthogonal Complements

Definition. Let W be a subspace of $\mathbb{R}^n$. The **orthogonal complement** of W is the subspace $W^\perp$ of all vectors that are orthogonal to W; that is

$$W^\perp := \{\mathbf{v} \in \mathbb{R}^n \ : \ \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\}.$$

Some helpful theorems:

- $(W^\perp)^\perp = W$
- $\mathrm{Nul}(A) = \mathrm{Col}(A^T)^\perp$
- $\mathrm{Nul}(A)^\perp = \mathrm{Col}(A^T)$
- $\mathrm{Nul}(A^T) = \mathrm{Col}(A)^\perp$

Theorem 43. *Let V be a subspace of $\mathbb{R}^n$. Then* $\dim V + \dim V^\perp = n$.

Coordinates

Standard basis (ε):

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Generally, if $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p$ are a basis B of vector space V, the coordinate vector of any vector $\mathbf{w}$ in V is:

$$\mathbf{w}_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_p \end{bmatrix}, \quad \text{if } \mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

This coordinate vector is unique!

Let $\mathcal{A} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ *be a basis of* $\mathbb{R}^3$.

(1) Find a 3×3-matrix B such that $\mathbf{v} = B\mathbf{v}_{\mathcal{A}}$ for all $\mathbf{v} \in \mathbb{R}^3$.

(2) Find a 3×3-matrix C such that $\mathbf{v}_{\mathcal{A}} = C\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^3$.

(1) Use the definition of coordinates to find:

B

$$\mathbf{v} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \boxed{\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}} v_{\mathcal{A}}$$

(2) Multiply B^{-1} on both sides:

C

$$\boxed{B^{-1}}\mathbf{v} = \mathbf{v}_{\mathcal{A}}$$

Change of Basis Matrix

Definition. Let $\mathcal{B}$ and $\mathcal{C}$ be two bases of $\mathbb{R}^n$. The **change of basis matrix** $I_{\mathcal{C},\mathcal{B}}$ is the matrix such that for all $\mathbf{v} \in \mathbb{R}^n$

$$I_{\mathcal{C},\mathcal{B}}\mathbf{v}_{\mathcal{B}} = \mathbf{v}_{\mathcal{C}}$$

Matrix allowing us to go from coordinates **mapped in B** to be **mapped onto C**

Theorem 45. *Let $\mathcal{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ be a basis of $\mathbb{R}^n$. Then*

$$I_{\mathcal{E}_n,\mathcal{B}} = \begin{bmatrix} \mathbf{b}_1 & \ldots & \mathbf{b}_n \end{bmatrix}$$

That is, for all $\mathbf{v} \in \mathbb{R}^n$,

$$\mathbf{v} = \begin{bmatrix} \mathbf{b}_1 & \ldots & \mathbf{b}_n \end{bmatrix} \mathbf{v}_{\mathcal{B}}.$$

How do we compute change of basis matrix:

$$I_{\mathcal{B},\mathcal{C}}?$$

What we know:

- $I_{En,B}$ = Matrix that maps **coordinates in B** onto **Standard**
- $I_{B,En}$ = Matrix that maps **coordinates in Standard** onto **B**

- $I_{En,C}$ = Matrix that maps **coordinates in C** onto **Standard**
- $I_{C,En}$ = Matrix that maps **coordinates in Standard** onto **C**

$$I_{\mathcal{B},\mathcal{E}_n} I_{\mathcal{E}_n,\mathcal{C}}$$

From right to left:
We map coordinates **from C into the standard** coordinate plane, **then**, we map the newly acquired **standard coordinates onto B's coordinate plane**

AKA: $I_{\mathcal{B},\mathcal{C}}$

Orthogonal and Orthonormal Bases

Definition. An **orthogonal basis** (an **orthonormal basis**) is an orthogonal set of vectors (an orthonormal set of vectors) that forms a basis.

Theorem 47. *Let $\mathcal{B} := (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ be an orthogonal basis of $\mathbb{R}^n$, and let $\mathbf{v} \in \mathbb{R}^n$. Then*

$$\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{b}_1}{\mathbf{b}_1 \cdot \mathbf{b}_1}\mathbf{b}_1 + \ldots + \frac{\mathbf{v} \cdot \mathbf{b}_n}{\mathbf{b}_n \cdot \mathbf{b}_n}\mathbf{b}_n.$$

When $\mathcal{B}$ is orthonormal, then $\mathbf{b}_i \cdot \mathbf{b}_i = 1$ for $i = 1, \ldots, n$.

Theorem 48. *Let $\mathcal{U} = (\mathbf{u}_1, \ldots, \mathbf{u_n})$ be an orthonormal basis of $\mathbb{R}^n$. Then*

$$h_{\mathcal{U},\mathcal{E}_n} = \begin{bmatrix} \mathbf{u}_1 & \ldots & \mathbf{u}_n \end{bmatrix}^T.$$

Why? An $n \times n$-matrix Q is **orthogonal** if $Q^{-1} = Q^T$

Small Group Time!

- Worksheets are organized by topic! Pick one or multiple to work through!

Midterm 3 Content (so far)

- Linear Transformation
- Coordinate Matrices
- Determinants
- Eigenvectors and eigenvalues
- Markov Matrices
- Diagonalization

Linear Transformation

Definition. Let V and W be vector spaces. A map $T : V \to W$ is a **linear transformation** if

$$T(a\mathbf{v} + b\mathbf{w}) = aT(\mathbf{v}) + bT(\mathbf{w})$$

for all $\mathbf{v}, \mathbf{w} \in V$ and all $a, b \in \mathbb{R}$.

$$T(\mathbf{0}_V) = T(0 \cdot \mathbf{0}_V) = 0 \cdot T(\mathbf{0}_V) = \mathbf{0}_W \rightsquigarrow T(\mathbf{0}_V) = \mathbf{0}_W$$

Check linearity with the zero vector!

Theorem 50. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there is a $m \times n$ matrix A such that*

- $T(\mathbf{v}) = A\mathbf{v}$, *for all* $\mathbf{v} \in \mathbb{R}^n$.
- $A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \end{bmatrix}$, *where* $(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)$ *is the standard basis of* $\mathbb{R}^n$.

Remark. We call this A the **coordinate matrix of** T with respect to the standard bases - we write $T_{\mathcal{E}_m, \mathcal{E}_n}$.

Coordinate matrices

Theorem 51. *Let V, W be two vector space, let $\mathcal{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ be a basis of V and $\mathcal{C} = (\mathbf{c}_1, \ldots, \mathbf{c}_m)$ be a basis of W, and let $T\colon V \to W$ be a linear transformation. Then there is a $m \times n$ matrix $T_{\mathcal{C},\mathcal{B}}$ such that*

- $T(\mathbf{v})_{\mathcal{C}} = T_{\mathcal{C},\mathcal{B}}\mathbf{v}_{\mathcal{B}}, \quad$ *for all* $\mathbf{v} \in V$.
- $T_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} T(\mathbf{b}_1)_{\mathcal{C}} & T(\mathbf{b}_2)_{\mathcal{C}} & \ldots & T(\mathbf{b}_n)_{\mathcal{C}} \end{bmatrix}$.

$$\begin{tikzcd}
\mathbf{v} : \text{vector in } V \arrow[r, "\text{apply } T"] \arrow[d, "\text{write in coordinates wrt } \mathcal{B}"'] & \text{vector in } W : T(\mathbf{v}) \arrow[d, "\text{write in coordinates wrt } \mathcal{C}"] \\
\mathbf{v}_{\mathcal{B}} : \text{coordinate vector in } \mathbb{R}^n \arrow[r, "\text{multiply by } T_{\mathcal{C},\mathcal{B}}"] & \text{coordinate vector in } \mathbb{R}^m : T_{\mathcal{C},\mathcal{B}}\mathbf{v}
\end{tikzcd}$$

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be such that $T(\mathbf{v}) = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \mathbf{v}$

$$\mathcal{B} := \left(\mathbf{b}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

Compute $T_{\mathcal{B},\mathcal{B}}$. Let $\mathbf{v} = \mathbf{b}_1 + \mathbf{b}_2$. Use $T_{\mathcal{B},\mathcal{B}}$ to compute $T(\mathbf{v})$

$$\mathbf{v}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$T_{\mathcal{B},\mathcal{B}} = \begin{bmatrix} T(\mathbf{b}_1)_{\mathcal{B}} & T(\mathbf{b}_2)_{\mathcal{B}} \end{bmatrix}$$

$$T(\mathbf{v})_{\mathcal{B}} = T_{\mathcal{B},\mathcal{B}} \mathbf{v}_{\mathcal{B}}$$

$$T(\mathbf{b}_1) = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2\mathbf{b}_1 + 0\mathbf{b}_2$$

$$T_{\mathcal{B},\mathcal{B}} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$T(\mathbf{b}_2) = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0\mathbf{b}_1 + 4\mathbf{b}_2$$

$$T(\mathbf{v}) = 2\mathbf{b}_1 + 4\mathbf{b}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

Theorem 52. *Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation and $\mathcal{A}$ and $\mathcal{B}$ be two bases of $\mathbb{R}^m$ and $\mathcal{C}, \mathcal{D}$ be two bases of $\mathbb{R}^n$. Then*

$$T_{\mathcal{C},\mathcal{A}} = I_{\mathcal{C},\mathcal{D}} T_{\mathcal{D},\mathcal{B}} I_{\mathcal{B},\mathcal{A}}.$$

Determinants (how to find them)

2x2: easy formula!

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Triangular: multiply all of the diagonal entries together

Otherwise: cofactor expansion

Note: if the matrix A is not invertible, det(A) = 0 ← this is the definition of a determinant!

Cofactor Expansion

Definition. Let A be an $n \times n$-matrix. The $(\mathbf{i}, \mathbf{j})$-**cofactor** of A is the scalar C_{ij} defined by

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$

Procedure for large matrices:

- Pick one row or one column to eliminate
- Go one by one in the other dimension (row or column) and ignore all the entries in that row + column
  - Calculate the cofactor
  - Find the determinant of the remaining matrix

This is very impractical for anything larger than 3x3!

Cofactor Expansion Example

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 2\cdot(-1)^{1+2}\cdot\begin{vmatrix} & - & \\ 3 & & 2 \\ 2 & & 1 \end{vmatrix} + (-1)\cdot(-1)^{2+2}\cdot\begin{vmatrix} 1 & & 0 \\ & + & \\ 2 & & 1 \end{vmatrix} + 0\cdot(-1)^{3+2}\cdot\begin{vmatrix} 1 & & 0 \\ 3 & & 2 \\ & - & \end{vmatrix}$$

$$= -2\cdot(-1) + (-1)\cdot 1 - 0 = 1$$

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 0\cdot(-1)^{1+3}\cdot\begin{vmatrix} & & + \\ 3 & -1 & \\ 2 & 0 & \end{vmatrix} + 2\cdot(-1)^{2+3}\cdot\begin{vmatrix} 1 & 2 & \\ & & - \\ 2 & 0 & \end{vmatrix} + 1\cdot(-1)^{3+3}\cdot\begin{vmatrix} 1 & 2 & \\ 3 & -1 & \\ & & + \end{vmatrix}$$

$$= 0 - 2\cdot(-4) + 1\cdot(-7) = 1$$

Properties of determinants

(Replacement) Adding a multiple of one row to another row *does not change* the determinant.
(Interchange) Interchanging two different rows *reverses the sign* of the determinant.
(Scaling) Multiplying all entries in a row by s, *multiplies* the determinant by s.

These three things also apply to the columns of a matrix!

Let A, B be two $n \times n$-matrices. Then $\det(AB) = \det(A)\det(B)$

If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$

Let A be an $n \times n$-matrix. Then $\det(A^T) = \det(A)$

Eigenvectors and Eigenvalues

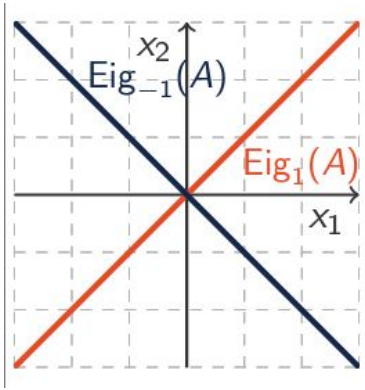
An **eigenvector** of A is a nonzero $\mathbf{v} \in \mathbb{R}^n$ such that

$$A\mathbf{v} = \lambda\mathbf{v}$$

eigenvalue

An eigenspace is all the eigenvectors associated with a specific eigenvalue.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



$$A\begin{bmatrix} x \\ x \end{bmatrix} = 1 \cdot \begin{bmatrix} x \\ x \end{bmatrix}$$

$$A\begin{bmatrix} -x \\ x \end{bmatrix} = -1 \cdot \begin{bmatrix} -x \\ x \end{bmatrix}$$

Eigenvectors are always linearly independent!

Calculating eigenvectors and eigenvalues

Theorem 59. *Let A be an $n \times n$ matrix. Then $p_A(t) := \det(A - tI)$ is a polynomial of degree n. Thus A has at most n eigenvalues.*

Definition. We call $p_A(t)$ the **characteristic polynomial** of A.

The roots of the characteristic polynomial are the eigenvalues

Let A be $n \times n$ matrix and let λ be eigenvalue of A. Then

$$\mathrm{Eig}_\lambda(A) = \mathrm{Nul}(A - \lambda I).$$

General algorithm: 1) find det(A-λI) and solve for λ
2) plug each eigenvalue back into A-λI
3) solve for the nullspace

Eigenvalue/eigenvector example

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 2 & 3 \\ 0 & 6-\lambda & 10 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (3-\lambda)(6-\lambda)(2-\lambda)$$

$\rightsquigarrow$ A has eigenvalues $2, 3, 6$. The eigenvalues of a triangular matrix are its diagonal entries.

$$\lambda_1 = 2: \qquad A - 2I = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 10 \\ 0 & 0 & 0 \end{bmatrix} \underset{RREF}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2.5 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \mathrm{Nul}(A-2I) = \mathrm{span}\left(\begin{bmatrix} 2 \\ -5/2 \\ 1 \end{bmatrix}\right)$$

$$\lambda_2 = 3: \qquad A - 3I = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 3 & 10 \\ 0 & 0 & -1 \end{bmatrix} \underset{RREF}{\rightsquigarrow} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \mathrm{Nul}(A-3I) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

$$\lambda_3 = 6: \qquad A - 6I = \begin{bmatrix} -3 & 2 & 3 \\ 0 & 0 & 10 \\ 0 & 0 & -4 \end{bmatrix} \underset{RREF}{\rightsquigarrow} \begin{bmatrix} 1 & \frac{-2}{3} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \mathrm{Nul}(A-6I) = \mathrm{span}\left(\begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix}\right)$$

Properties of Eigenvalues and Eigenvectors

For a 2x2 matrix:

$$p(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A)$$

Multiplicity:

- **Algebraic** multiplicity is the multiplicity of λ in the characteristic polynomial
- **Geometric** multiplicity is the dimension of the eigenspace of λ

Trace: the sum of the diagonal entries of a matrix

- Tr(A) = sum of all eigenvalues
- det(A) = product of all eigenvalues

Markov Matrices

$$\begin{bmatrix} 0 & .25 & .4 \\ 1 & .25 & .2 \\ 0 & .5 & .4 \end{bmatrix}$$

Definition: a square matrix with non-negative entries where the sum of terms in each column is 1

A **probability vector** has entries that add up to 1

The λ of a Markov Matrix:

- 1 is always an eigenvalue, and the corresponding eigenvector is called **stationary**
- All other $|\lambda| \leq 1$

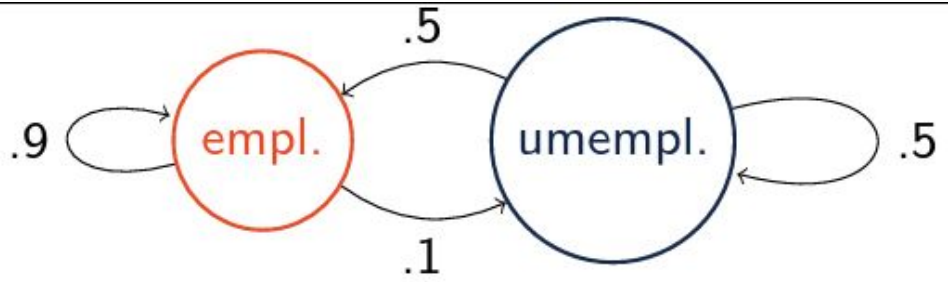
Why is a Markov Matrix useful?

Theorem 65. *Let A be an $n \times n$-Markov matrix with only positive entries and let $\mathbf{z} \in \mathbb{R}^n$ be a probability vector. Then*

$$\mathbf{z}_\infty := \lim_{k \to \infty} A^k \mathbf{z} \text{ exists,}$$

and $\mathbf{z}_\infty$ is a stationary probability vector of A (ie. $A\mathbf{z}_\infty = \mathbf{z}_\infty$).

This basically says you can left multiply A with **z** infinitely and you will get a stationary probability vector (steady state)



x_t: % of population employed at time t

y_t: % of population unemployed at time t

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} .9x_t + .5y_t \\ .1x_t + .5y_t \end{bmatrix} = \begin{bmatrix} .9 & .5 \\ .1 & .5 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

How to approach a Markov Matrix problem

1. Write out the Markov Matrix A. If it helps, make a graph like on the previous slide.
2. Determine what the question is asking you to solve for. Steady state? Intermediate state?
3. Write the probability vector of what you know of the initial state, if possible.
4. To solve for the **steady state**: Find A-1*I and solve for the nullspace, then find the probability vector in the nullspace
5. To solve for an **intermediate state**: multiply the initial state vector by the Markov matrix the appropriate number of times.

Diagonalization

$$P = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$$

v are eigenvectors

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

For a matrix A to be diagonalizable:

- A must be square
- A must have as many unique eigenvectors as rows/columns (i.e. it has an eigenbasis)
- $A = PDP^{-1}$

Observe that

$$A = PDP^{-1} = I_{\mathcal{E}_n,\mathcal{B}} D I_{\mathcal{B},\mathcal{E}_n}$$

Where B is the eigenbasis → diagonalizing is a base change to the eigenbasis

In-Person Resources

CARE Drop-in tutoring:
7 days a week on the 4th floor of Grainger Library!
Sunday - Thursday 12pm-10pm
Friday & Saturday 12-6pm

Course Office hours:
TAs: Monday - Thursday 5-7pm in English Building 108
Instructors: Chuang MW 4-5PM in CAB 233
Leditzky MW 2:30-3:30PM in CAB 39
Luecke Tu 1:30-3:30PM in Altgeld 105

Subject	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
Math 257	3pm-9pm	2pm-4pm 8pm-10pm	2pm-6pm 8pm-10pm	2pm-5pm 6pm-10pm	2pm-9pm	1pm-6pm	12pm-6pm