The Grainger College of Engineering
Center for Academic Resources in Engineering

MATH 241

Midterm 2 Review

Keep in mind that this presentation was created by CARE tutors, and while it is thorough, it is not comprehensive.

QR Code to the Queue

The queue contains the worksheet and the solution to this review session

Partial Derivatives

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \qquad f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

$$f(x,y) \quad \Rightarrow \quad f_x(x,y) = \frac{\partial f}{\partial x} \;\; \& \;\; f_y(x,y) = \frac{\partial f}{\partial y}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(f_x) = (f_x)_x = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(f_y) = (f_y)_y = f_{yy}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(f_x) = (f_x)_y = f_{xy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(f_y) = (f_y)_x = f_{yx}$$

Linear Approximation

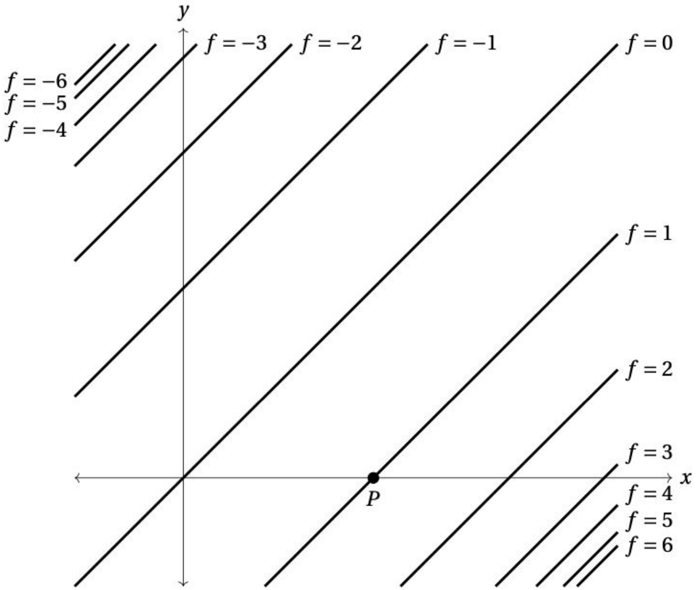
- If $z = f(x, y)$ and f is **differentiable** at (a, b), then the value of f(m, n) can be approximated by

$$f(\mathrm{m}, \mathrm{n}) \approx L(\mathrm{m}, \mathrm{n})$$

$$L(\mathrm{m}, \mathrm{n}) = \mathrm{f}(\mathrm{a}, \mathrm{b}) + \mathrm{f}_{\mathrm{x}}(\mathrm{a}, \mathrm{b}) \cdot (\mathrm{m} - \mathrm{a}) + \mathrm{f}_{\mathrm{y}}(\mathrm{a}, \mathrm{b}) \cdot (\mathrm{n} - \mathrm{b})$$

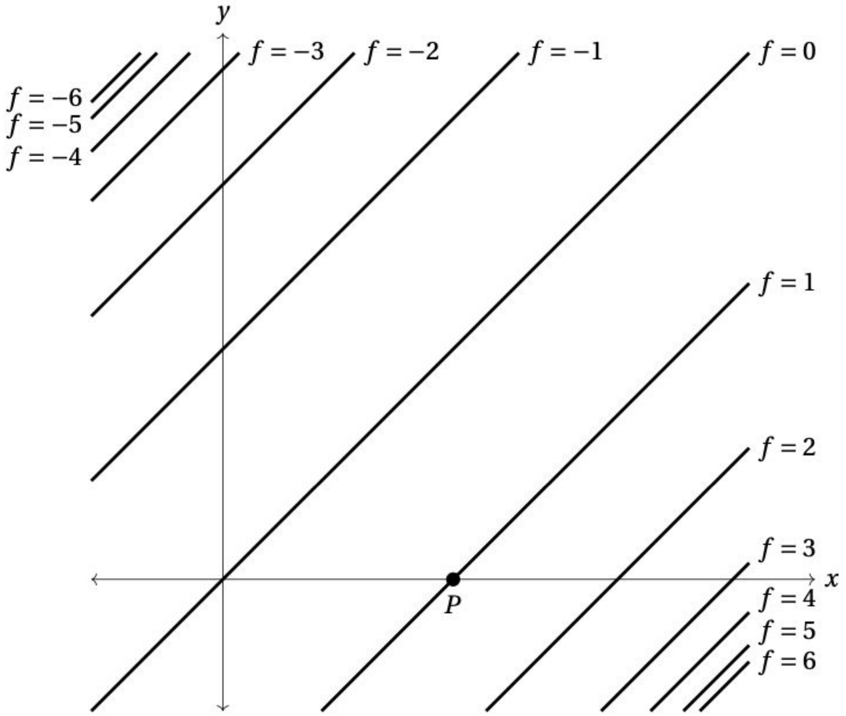
Example Question #1

- A contour map for a function f of x, y, and a point P in the plane are given below. Determine if the following quantities are negative, zero, or positive: $f_x(P)$, $f_{xx}(P)$, $f_{xy}(P)$



Example Solution #1

- $f_x(P)$: positive
- $f_{xx}(P)$: positive
- $f_{xy}(P)$: negative



Limits and Continuity

- When computing multivariable limits,
  - Check **multiple paths** (lines and power functions) to see if there are conflicting values. If so, limits DNE
  - **Factor** (difference of squares)
  - Use **polar coordinates**
  - Try **squeeze theorem**

Example Question #2

- Compute the following limits

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} \qquad \lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8} \qquad \lim_{x\to 0} x^3 \cos\left(\frac{2}{x}\right) \qquad \lim_{(x,y)\to(-1,0)} \frac{x^2+xy+3}{x^2y-5xy+y^2+1}$$

- Determine whether the following function is continuous at (0, 0)

$$f(x,y) = \begin{bmatrix} \dfrac{xy}{x^2+xy+y^2}, (x,y) \neq (0,0) \\ 0 \qquad\qquad\quad , (x,y) = (0,0) \end{bmatrix}$$

Example Solution #2

- Compute the following limits

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$$ **= 0** (Use polar coordinates)

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$$ **= DNE** (Check $x = y^4$ and $x = -y^4$)

$$\lim_{x\to0} x^3 \cos\left(\frac{2}{x}\right)$$ **= 0** (Squeeze Theorem)

$$\lim_{(x,y)\to(-1,0)} \frac{x^2+xy+3}{x^2y-5xy+y^2+1}$$ **= 4** (Plug in (–1, 0) directly)

Example Solution #2

- Determine whether the following function is continuous at (0, 0)

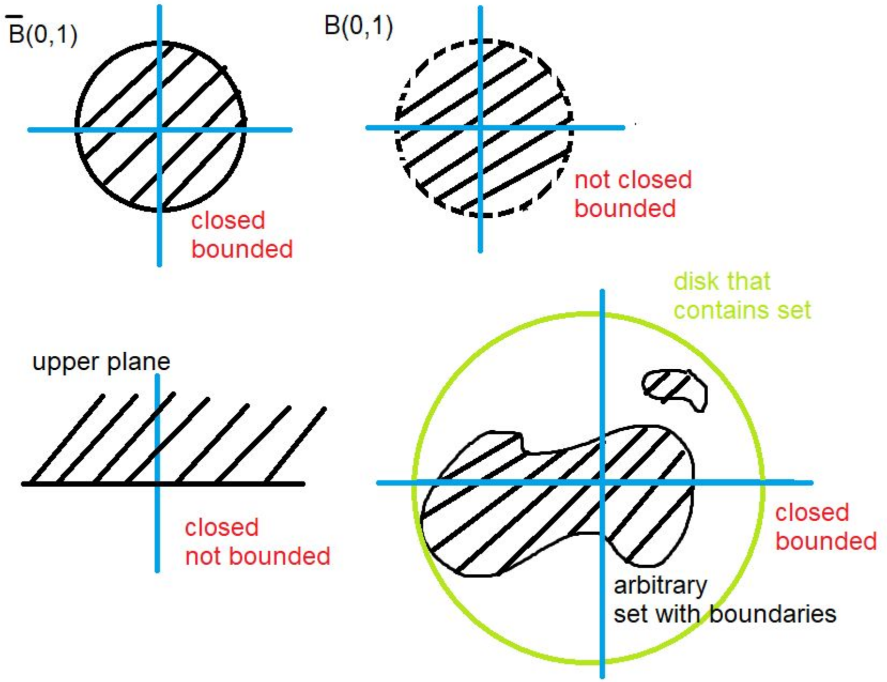
$$f(x,y) = \begin{bmatrix} \dfrac{\mathrm{xy}}{\mathrm{x}^2 + \mathrm{xy} + \mathrm{y}^2}, (x,y) \neq (0,0) \\ 0 \qquad\qquad\quad , (x,y) = (0,0) \end{bmatrix}$$

On line y = x, f(x, y) = 1/3 at any point except (0, 0). Since there is a discontinuity at (0, 0), the function is not continuous.

Extreme Value Theorem

If f(x,y) is continuous on a closed and bounded set D, then it is guaranteed that f has an absolute minimum and maximum value

- The absolute min and max will either occur at the critical points of f, or on the endpoints of the boundary D



https://math.stackexchange.com/questions/1190640/what-is-the-difference-between-closed-and-bounded-in-terms-of-domains

Example Question #3

- Consider the function $f = x^3 + y^3 + 3xy$. If the critical points of f are (0, 0) and (–1, –1), classify them into local mins, maxes, and saddle points.

Example Solution #3

- Consider the function $f = x^3 + y^3 + 3xy$. If the critical points of f are (0, 0) and (–1, –1), classify them into local mins, maxes, and saddle points.

$$f_x = 3x^2 + 3y \quad f_y = 3y^2 + 3x \quad f_{xx} = 6x \quad f_{yy} = 6y$$

$$f_{xy} = f_{yx} = 3$$

$$\text{At } (0, 0), D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} = -9 \rightarrow \text{ Saddle Point}$$

$$\text{At } (-1, -1), D = \begin{vmatrix} -6 & 3 \\ 3 & -6 \end{vmatrix} = 27 \rightarrow \text{ Because } f_{xx} = -6 < 0 \rightarrow \text{ Local Max}$$

Gradient and Directional Derivatives

$$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

- The gradient will always point **perpendicular to the level curves/surfaces of f**
- ∇f = 0 at a local minimum/maximum

$$D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

- Tells you how the function f changes along the vector u

Lagrange Multiplier

- Solve the following system of equations for λ (Lagrange Multiplier)
  - Where f is the function, and g is the constraint

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$
$$g(x, y, z) = k$$