# Center for Academic Resources in Engineering (CARE)
# Peer Exam Review Session

### Math 231 – Calculus II

### Midterm 2 Worksheet Solutions

---

*The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.*

---

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: February 29th, 6-7:30pm Sofi and Pranav

Session 2: March 3rd, 2:30-4pm Soundarya and Bella

Can't make it to a session? Here's our schedule by course:

https://care.grainger.illinois.edu/tutoring/schedule-by-subject

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/844
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

**Please be sure to follow the above steps to add yourself to the Queue.**

Good luck with your exam!

1. Determine whether the integral converges or diverges. If so, what value does the integral converge to?

$$\int_5^\infty \frac{1}{n(\ln(n))^2}$$

$= \int_5^\infty \frac{1}{x(\ln(x))^2} dx \; [u = \ln(x),\, du = \frac{1}{x}]$

$= \frac{-1}{\ln(x)}|_5^\infty = \frac{1}{\ln(5)} \rightarrow$ so the integral converges.

2. We want to evaluate

$$\int_0^2 x^2 dx$$

using a Riemann sum of $n = 3$ terms. Let us define $L_3$ as the Riemann Sum if we choose the left endpoints and $R_3$ if we choose the right endpoints. Then:

(a) $L_3 = \frac{9}{32}$, $R_3 = \frac{15}{32}$

(b) $L_3 = \frac{40}{27}$, $R_3 = \frac{112}{27}$

(c) $L_3 = \frac{15}{81}$, $R_3 = \frac{41}{81}$

(d) $L_3 = \frac{7}{32}$, $R_3 = \frac{55}{36}$

(e) $L_3 = \frac{53}{36}$, $R_3 = \frac{55}{36}$

We have $\Delta x = \frac{2}{3}$

Choose the left-hand endpoints

$x_1 = 0$, $x_2 = \frac{2}{3}$, $x_3 = \frac{4}{3}$

$$L_3 = f(x_1)\delta x + f(x_2)\Delta x + f(x_3)\Delta x$$

$$0^2\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2\left(\frac{2}{3}\right) + \left(\frac{4}{3}\right)^2\left(\frac{2}{3}\right)$$

$$\boxed{L_3 = \frac{40}{27}}$$

Now, choose the right endpoints

$x_1 = \frac{2}{3}$, $x_2 = \frac{4}{3}$, $x_3 = 2$

$$R_3 = f(x_1)\Delta x + f(x_2)\delta x + f(x_3)\Delta x$$

$$\left(\frac{2}{3}\right)^2\left(\frac{2}{3}\right)+\left(\frac{4}{3}\right)^2\left(\frac{2}{3}\right)+(2)^2\left(\frac{2}{3}\right)$$

$$\boxed{R_3=\frac{112}{27}}$$

3. Test the following improper integral for convergence. If it converges, find the value

$$\int_0^\infty e^{-\frac{4}{3}y}dy$$

(a) Diverges
(b) Converges, value is $-\frac{3}{4}$
(c) Converges, value is $\frac{4}{3}$
(d) Converges, value is $\frac{3}{4}$
(e) None of the above

This is an improper integral, so we need to check for convergence.

$$\lim_{t\to\infty}\int_0^t e^{-\frac{4}{3}y}dy$$

$$\lim_{t\to\infty}\left(\frac{-3}{4}e^{\frac{-4}{3}y}\right)\Big|_0^t$$

$$\lim_{t\to\infty}\left(\left(\frac{-3}{4}e^{-\frac{4}{3}t}\right)-\left(-\frac{3}{4}e^0\right)\right)$$

$$\lim_{t\to\infty}\frac{-3}{4e^{\frac{4}{3}t}}+\frac{3}{4}$$

$$\frac{-3}{\infty}+\frac{3}{4}=\boxed{\frac{3}{4}}$$

4. Determine if the following integral is convergent or divergent:

$$\int_2^\infty \frac{\cos^2 x}{x^2}dx$$

To solve this we must use the comparison test. Note that the numerator is bounded by:

$$0\le\cos^2 x\le 1$$

Therefore the behavior of the denominator will determine the convergence/divergence. The integral

$$\int_2^\infty \frac{1}{x^2} dx$$

must converge via the $p$-test since $p = 2 > 1$. Putting these two facts together, we see that

$$\frac{\cos^2 x}{x^2} \leq \frac{1}{x^2}$$

, as the numerator of $\cos^2 x$ will always be $\leq 1$. Since $\int_2^\infty \frac{1}{x^2} dx$ converges, then our original integral $\int_2^\infty \frac{\cos^2 x}{x^2} dx$ must converge.

5. Evaluate the following integral using a Trapezoidal Riemann sum with 4 equal intervals:

$$\int_0^4 \frac{x^{0.5}}{2} + 1$$

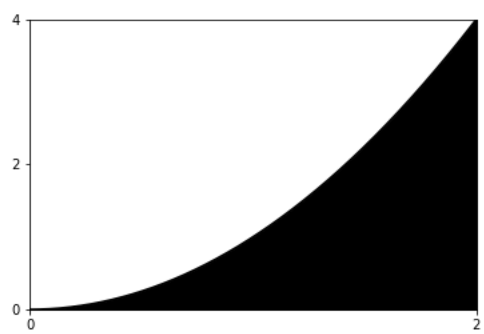
First we solve for $\Delta x$: $\Delta x = \frac{b-a}{n} = \frac{4-0}{4} = 1$

Using the formula for Trapezoidal Riemann sums: $\frac{1}{2}\Delta x(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4))$ where $x_i = a + i\Delta x$

Plugging into the formula:

$T_n = \frac{1}{2}(1)(f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)) = \frac{1}{2}(1 + 2(1.5) + 2(1.7) + 2(1.9) + 2) = 6.6$

6. Find the $M_x$, $M_y$, and the centroid of $y = x^2$ with density $\lambda$ on $x \in [0, 2]$.



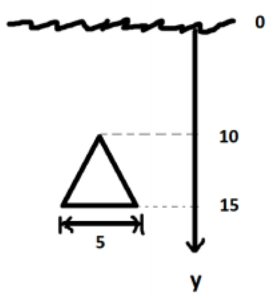

$mass = \lambda \int_0^2 x^2 dx = \frac{\lambda}{3} x^3|_0^2 = \frac{8\lambda}{3}$

$M_x = \frac{\lambda}{2}\int_0^2 (x^2)^2 dx = \frac{\lambda}{10}x^5|_0^2 = \boxed{\frac{16\lambda}{5}}$

$M_y = \lambda \int_0^2 x(x^2)dx = \frac{\lambda}{4}x^4|_0^2 = \boxed{4\lambda}$

Centroid (x,y) $= (\frac{M_y}{mass}, \frac{M_x}{mass}) = \boxed{(\frac{3}{2}, \frac{6}{5})}$

7. Determine the hydrostatic force on the triangle given the density of water $\rho = 1000\text{kg/m}^3$ with a depth $y$ and $g = 9.8\text{m/s}^2$.

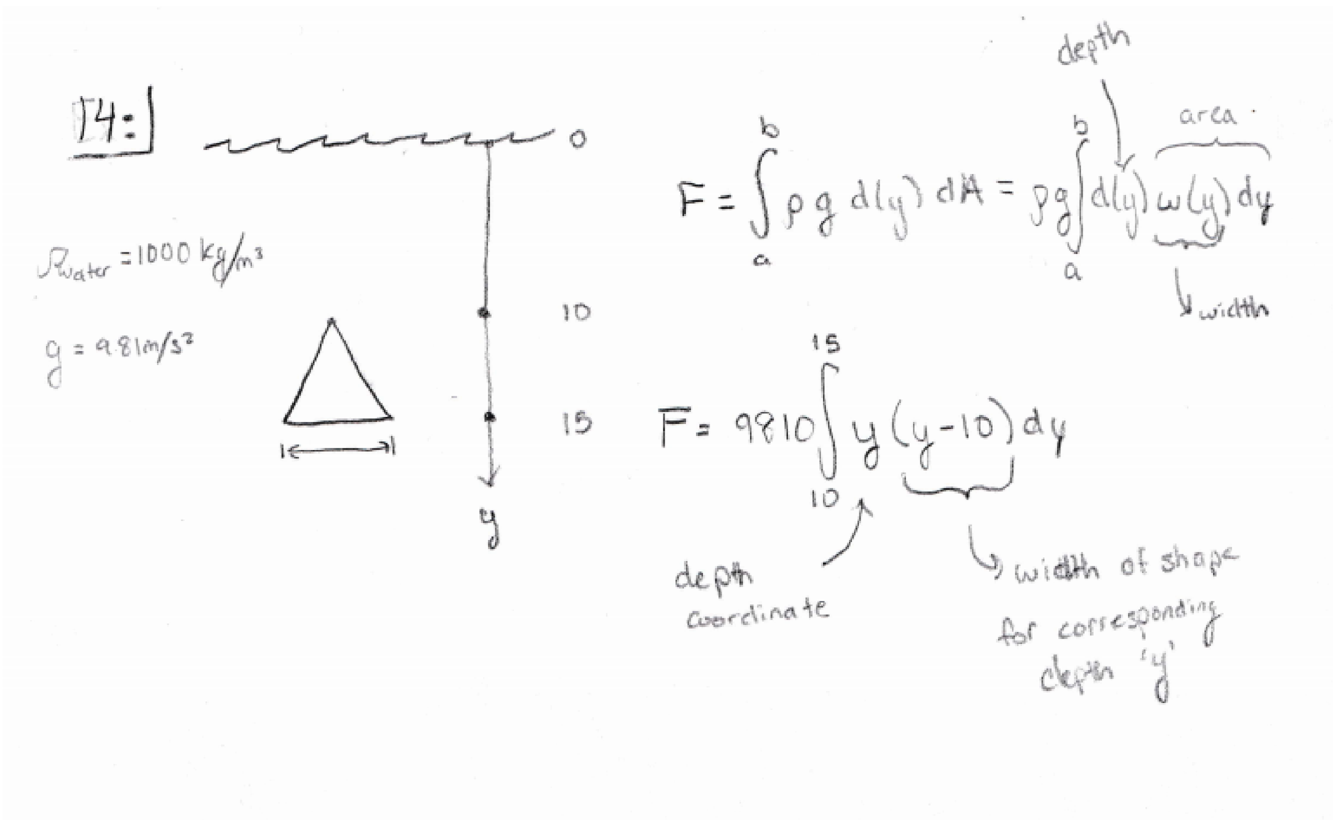



See image below for what each term means within the integral

$F = \int_a^b \rho g d(y) dA = \rho g \int_a^b d(y)w(y)dy$

$F = 9810\int_{10}^{15} y(y-10)dy$

$\boxed{\text{F} = 1635000N}$




8. Consider the curve y = 5ln(x) between the points (1,0) and (e,5).

A. SET UP, BUT DO NOT EVALUATE, a dx-integral which represents the arc length of the curve.
B. SET UP, BUT DO NOT EVALUATE, a dy-integral which represents the arc length of the curve.
C. SET UP, BUT DO NOT EVALUATE, a definite integral which represents the surface area of the surface obtained by rotating the curve around the line y =10.

A. $ds = \sqrt{1+(\frac{dy}{dx})^2}dx$, $y' = \frac{5}{x}$, $ds = \int_1^e \sqrt{1+(\frac{5}{x})^2}$

B. $y = 5\ln(x)$, $e^{\frac{y}{5}} = e^{\ln(x)}$, $x = e^{\frac{y}{5}}$, $x' = \frac{e^{\frac{y}{5}}}{5}$, $ds = \int_0^5 \sqrt{1+(\frac{e^{\frac{y}{5}}}{5})^2}dy$

C. $SA = \int 2\pi y ds$, $ds = \sqrt{1+(\frac{e^{\frac{y}{5}}}{5})^2}dy$, $SA = 2\pi \int_0^5 (10-y)\sqrt{1+(\frac{e^{\frac{y}{5}}}{5})^2}dy$

9. The profile $y = \sqrt{4-x^2}$ on the interval $x \in [-1, 1]$ is revolved around the $x$-axis. Find the surface area of this surface.
Use the following formula to find the surface area of an arc rotated about the $x$-axis:

$$S = \int 2\pi y ds$$

$$\text{where } ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

We choose to integrate with respect to $x$ since we are given that interval.

$$S = \int_{-1}^{1} 2\pi y \sqrt{1 + \left(\frac{-x}{\sqrt{4-x^2}}\right)^2}\, dx$$

We can simplify the square root term from the $ds$.

$$\sqrt{1 + \left(\frac{-x}{\sqrt{4-x^2}}\right)^2} = \sqrt{1 + \frac{x^2}{4-x^2}} = \sqrt{\frac{4-x^2}{4-x^2} + \frac{x^2}{4-x^2}} = \sqrt{\frac{4}{4-x^2}} = \frac{2}{\sqrt{4-x^2}}$$

$$S = \int_{-1}^{1} 2\pi y \frac{2}{\sqrt{4-x^2}} dx$$

We can then write $y$ in terms of $x$ and simplify.

$$S = \int_{-1}^{1} 2\pi \sqrt{4-x^2} \frac{2}{\sqrt{4-x^2}} dx$$

$$S = \int_{-1}^{1} 4\pi dx = \boxed{8\pi}$$

10. Compute the arc length of the function $y = 1 + 2x^{\frac{3}{2}}$ between $x = 0$ and $x = 1$

(a) $\frac{14}{9}$

(b) $\frac{10}{9}$

(c) $\frac{2}{9}\sqrt{10}$

(d) $\frac{2}{27}(10\sqrt{10} + 1)$

(e) None of the above

$$ds = \sqrt{1 + (\frac{dy}{dx})^2}$$

$$y = 1 + 2x^{\frac{3}{2}}, \frac{dy}{dx} = 3x^{\frac{1}{2}}$$

$$S = \int_0^1 \sqrt{1 + (3x^{\frac{1}{2}})^2} dx$$

$$S = \int_0^1 \sqrt{1 + 9x} dx$$

$u = 1 + 9x$, $du = 9dx$, $dx = \frac{1}{9}du$

The u-bounds are: $u = 1 + 9(1) = 10$, $u = 1 + 9(0) = 1$

$$S = \sqrt{u}\frac{1}{9}du$$

$$S = \frac{1}{9}(\frac{2}{3})u^{\frac{3}{2}}|_1^{10}$$

$$(\frac{2}{27})(10^{\frac{3}{2}} - 1^{\frac{3}{2}}$$

$$S = \left(\frac{2}{27}\right)(10\sqrt{10} - 1)$$

The answer is therefore **(e)**.