MATH 257 Exam 1 CARE Review

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Topic Summary

- Linear systems
  - Solving systems with matrices
- Reduced row echelon form
  - Pivot columns: basic and free variables
- Elementary matrices
  - Elementary row operations
- Vectors and spans
- Matrix operations
  - Addition, subtraction, scalar multiplication, linear combinations
  - Transposition
- Matrix multiplication
  - Properties of matrix multiplication
- Matrix inverses
  - What matrices are invertible?

Linear Systems

$$a_1x_1 + \ldots + a_nx_n = b$$

and matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Linear systems must have either:

1. One unique solution
2. Infinite solutions
3. No solutions

Equivalent linear systems have the same set of solutions.

You can represent a linear system with matrices…

linear system	coefficient matrix	augmented matrix

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Column vector

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Row vector

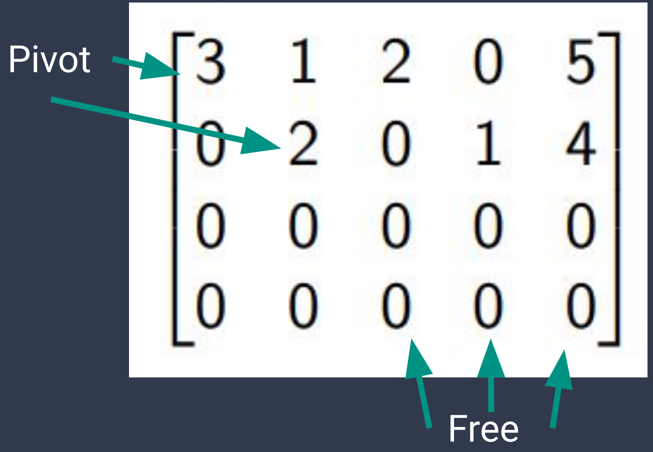
We often define a matrix in terms of its columns or its rows:

$\mathbf{a}_n$ are all column vectors

$$A := \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \quad or \quad A := \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \vdots \\ \mathbf{R}_m \end{bmatrix}$$

$\mathbf{R}_m$ are all row vectors

Echelon Forms



$$\begin{bmatrix} 3 & 1 & 2 & 0 & 5 \\ 0 & 2 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \end{bmatrix}$$

Row Echelon Form (REF):

1. All nonzero rows above rows of all zeros
2. Leading entry (leftmost nonzero number) is strictly to the right of the leading entry of the row above

Reduced Row Echelon Form (RREF):

3. Leading entries of nonzero rows are all 1
4. Each leading entry is the only nonzero entry in the column

Gaussian Elimination (for a general solution)

It’s an algorithm!

1. Write down the augmented matrix.
2. Find the RREF of the matrix.
3. Write down linear equations based on the RREF.
4. Express pivot variables in terms of free variables (unless there are no free variables).
5. Solve only if there are no **free variables** and the matrix is **consistent**. (This means the solution is unique!)

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \leftarrow \qquad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{array}\right] \leftarrow$$

$0 = 0$ ✓

Dependent

AKA consistent

$0 = 2$ ✗

Inconsistent

Inconsistent systems have no solutions.

Elementary Row Operations

Elementary operations do not change the solution set of a system.

There are three kinds:

1. Replacement ($R_1 \rightarrow R_1 + a*R_2$)
2. Scaling ($R_1 \rightarrow a*R_1$)
3. Interchange ($R_1 \rightarrow R_2$)

All elementary operations are reversible. Two matrices are **row equivalent** if elementary operations can turn one into the other.

Matrix Operations

a) **The sum of** $A + B$ is

$$\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

b) **The product** cA for a scalar c is

$$\begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}$$

Addition: only defined for matrices with the **same dimensions**

Subtraction: the same as addition

Scalar multiplication: every entry is multiplied by the **scalar**

- Scalar = any real number

Linear combinations: any mixture of scalar multiplication and addition/subtraction of matrices

- **span(a,b)** is a set of ALL the possible linear combinations of **a** and **b**

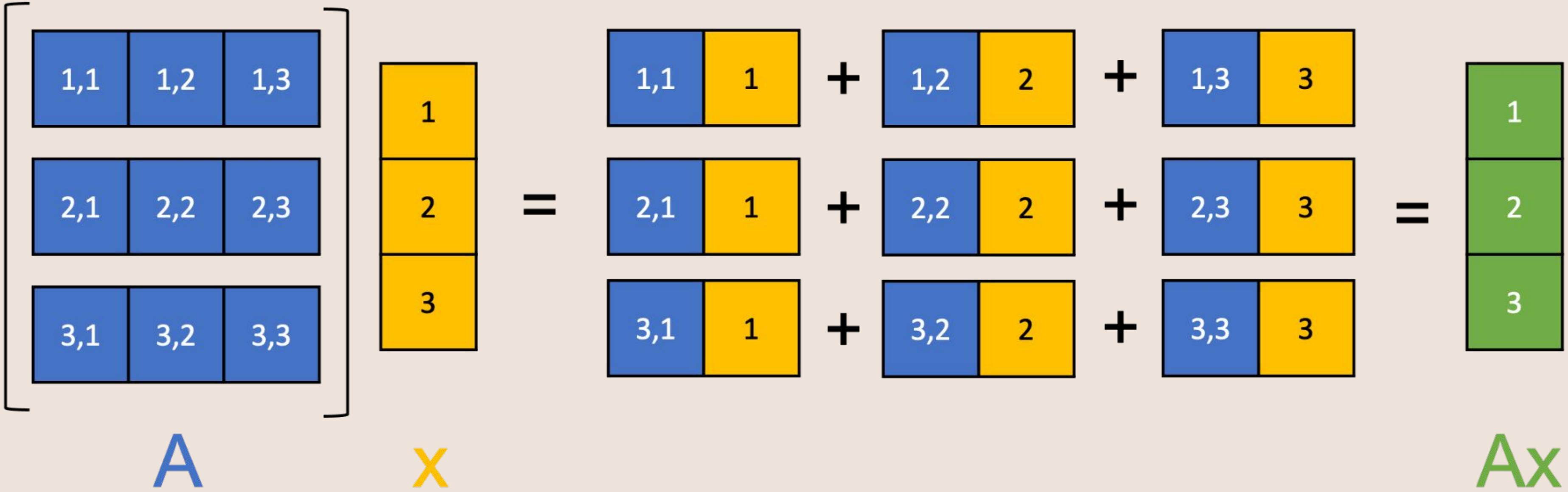
Matrix Operations (cont.)

Transpose: switch rows and columns

2	4	-1
-10	5	11
18	-7	6

2	-10	18
4	5	-7
-1	11	6

Matrix-vector multiplication: $A\mathbf{x} = x_1\mathbf{a_1} + x_2\mathbf{a_2} + \dots + x_n\mathbf{a_n}$ which means you multiply the **entries** of the vector with the **columns** of the matrix



Matrix-vector multiplication

- The number of entries in **x** must match the number of columns in A

Matrix multiplication

Only defined for two matrices A and B if

- A has the dimensions *m* x *n* and B has the dimensions *n* x *p*
- A^k (exponent) is only defined for a **square** matrix

Each entry of AB is a linear combination of a **row of A** with a **column of B.**

$$AB = \begin{bmatrix} \mathbf{R}_1\mathbf{C}_1 & \dots & \mathbf{R}_1\mathbf{C}_p \\ \mathbf{R}_2\mathbf{C}_1 & \dots & \mathbf{R}_2\mathbf{C}_p \\ \mathbf{R}_m\mathbf{C}_1 & \dots & \mathbf{R}_m\mathbf{C}_p \end{bmatrix} \quad \text{and} \quad (AB)_{ij} = \mathbf{R}_i\mathbf{C}_j = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

Properties of Matrix Multiplication

(a) $A(BC) = (AB)C$ *(associative law of multiplication)*

(b) $A(B + C) = AB + AC$, $(B + C)A = BA + CA$ *(distributive laws)*

(c) $r(AB) = (rA)B = A(rB)$ *for every scalar r,*

(d) $A(rB + sC) = rAB + sAC$ *for every scalars r, s (linearity of matrix multiplication)*

(e) $I_m A = A = AI_n$ *(identity for matrix multiplication)*

Transpose Theorem: $(AB)^T = B^T A^T$

Matrix multiplication is NOT COMMUTATIVE: AB ≠ BA

Elementary Matrices

Identity Matrices

$1 \times 1 \quad [1]$

$$2 \times 2 \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$3 \times 3 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

etc.

Any matrix that can be form from the identity matrix with **one** elementary row operation.

Ex.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{R_2 \to 3R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{R_2 \leftrightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Elementary Matrices (cont.)

Multiplying an elementary matrix (E) with another matrix (A) is the same as performing the elementary row operation on A.

This means you can represent putting a matrix in RREF as a sequence of matrix multiplications:

$E_n \ldots E_2 E_1 A = B$ where A is the original matrix and B is the RREF form

Matrix Inverses

Determinants:

$$\frac{1}{ad - bc}$$

For the matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Definition of an inverse:

$$AC = I_n$$

Requirements for a matrix to be invertible:

1. It has to be square
2. The determinant of the matrix cannot be 0 or
3. The RREF of A is the identity matrix or
4. A has as many pivots as columns/rows

Statements 2, 3, and 4 mean the same thing.

Calculating an Inverse

For 2x2:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Elementary Matrix strategy:

$$A^{-1} = E_m E_{m-1} \dots E_1 = E_m E_{m-1} \dots E_1 I_n.$$

OR: set up an augmented matrix with the identity and reduce to RREF

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{array}\right]$$

Works for any square matrices of any size

Properties of Matrix Inverses

(a) A^{-1} *is invertible and* $\left(A^{-1}\right)^{-1} = A$ (*i.e.* A *is the inverse of* A^{-1}).

(b) AB *is invertible and* $(AB)^{-1} = B^{-1}A^{-1}$.

(c) A^T *is invertible and* $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

Inverses are unique! Every invertible matrix only has one inverse.

Multiplying by a matrix inverse is the closest we get to dividing matrices.

Theorem 14. *Let* A *be an invertible* $n \times n$ *matrix. Then for each* $\mathbf{b}$ *in* $\mathbb{R}^n$, *the equation* $A\mathbf{x} = \mathbf{b}$ *has the unique solution* $\mathbf{x} = A^{-1}\mathbf{b}$.

Python Coding Tips

Remember to **import** numpy and math!
import numpy as np
from math import *

Check for **syntax errors** (missing parentheses and brackets, spelling)

- Read your error message! It usually tells you exactly where it went wrong

You have to use **np.** or **np.linalg.** for most functions

Study **coding problems from the homework** (hint: they tend to pull questions from there!)

Python Functions to know

Useful functions to know:

np.array([[1, 1, 1], [2, 2, 2]]) → $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$

np.solve(a, b) → solves a system where a is the coefficient matrix and b is the scalars on the right side of the =

np.inv(a) → gives you the inverse if a is invertible

Ways to multiply matrices:

a @ b ← this is always matrix multiplication

a * b ← don't use this unless a or b is a scalar

np.dot (a, b) ← gives the dot product

Questions?

Join the queue to see the worksheet!