QMA The normalized eigenfunctions of the Hydrogen-atom Hamiltonian H are

$$\psi_{n,l,m}(r,\theta,\phi) = \sqrt{\left(\frac{2}{na_0}\right)^3 \left(\frac{(n-l-1)!}{2n(n+l)!}\right)} e^{-\rho}\rho^l L^{2l+1}_{n-l-1}(\rho) Y^l_m(\theta,\phi).$$

Here a_0 is the Bohr radius, $\rho = 2r/(na_0)$, Y^l_m is the normalized spherical harmonic, and the generalized Laguerre polynomial has $L^\alpha_n(0) = (n+\alpha)!/(n!\alpha!)$.

Your task is to evaluate the effect of a perturbation $\Delta H = -A\delta^3(\mathbf{r})$ (a contribution to the Lamb shift) on the energy eigenvalues associated with the above eigenfunctions. You may ignore the proton spin.

a) For the general eigenstate, write down the expression giving the first-order energy shift due to the perturbation. State for which values of n, l, and m the first-order shift is nonzero.

b) After incorporating spin-orbit corrections, the hydrogen energy eigenvalues depend on n and the total angular momentum j, but not separately on the orbital number l. For a given n, the states with the smallest j have the lowest energy. Incorporating this knowledge—but ignoring the effect of ΔH—how many degenerate states are there for the lowest $n = 2$ level, and what is the associated value of j?

c) Write down the wavefunction, including the spin state of the electron, for the lowest $n = 2$ level states you found in (b). Write your answer as as linear combination of terms of the form $\psi_{nlm}|\pm\rangle$ where $|\pm\rangle$ is the electron spin state.

d) Using degenerate perturbation theory, find the splitting caused by the perturbation (to first order in A) between the degenerate levels you found in (b) .

Useful: Clebsch-Gordan coefficients for general j_1 and $j_2 = 1/2$

$\langle j_1, m_1, j_2 = 1/2, m_2 | j, m\rangle$

	$m_2 = 1/2$	$m_2 = -1/2$
$j = j_1 + 1/2$	$\sqrt{\frac{j_1+m+1/2}{2j_1+1}}$	$\sqrt{\frac{j_1-m+1/2}{2j_1+1}}$
$j = j_1 - 1/2$	$-\sqrt{\frac{j_1-m+1/2}{2j_1+1}}$	$\sqrt{\frac{j_1+m+1/2}{2j_1+1}}$