CMB You are a space explorer traveling in your spaceship of mass m. Inspired by the Nobel-prize winning discovery of the supermassive black hole Sgr A* at the center of the Milky Way with $M = 4.2 \times 10^6$ solar masses, you wish to investigate the central region of our galaxy. Originally you are sufficiently far away from the black hole such that gravitational fields are weak, and Newtonian gravity describes your dynamics.

a) Derive the Lagrangian $L(r, \theta, \phi)$ describing the spaceship in the gravitational field of M within Newtonian gravity. Here (r, θ, ϕ) are spherical coordinates. Suppose that the orbital angular momentum $\mathbf{L} = l_z \mathbf{e}_z$ is aligned with the z-axis and that the spaceship is moving in the equatorial plane $\theta = \pi/2$.

b) From $L(r, \theta, \phi)$ obtain the equations of motion for $r(t)$ and $\phi(t)$.

c) Derive the energy and angular-momentum conservation laws for this system. Show that for a given angular momentum l_z the energy conservation equation can be written in the form

$$\frac{E}{m} = \frac{1}{2}\dot{r}^2 + V_{\text{eff}} = \text{const.},$$

where you should find $V_{\text{eff}}(r)$.

d) Your spaceship is in a circular orbit around the black hole. For a given value of $|\mathbf{L}|$ what is the orbital radius r_c?

e) Suppose your orbit is perturbed in a manner that does not alter $\mathbf{L}$. Determine if your orbit is stable.

Now suppose that you are close to Sgr A* where relativistic effects become important. In particular, it is possible that *light rays* can be trapped in a circular orbit. For a light ray, the effective potential is modified and the relevant energy equation becomes

$$\frac{1}{2}\dot{r}^2 + \frac{|\mathbf{L}|^2}{2r^2} - \frac{GM|\mathbf{L}|^2}{r^3} = \text{const.}$$

f) Show that circular orbits for light exist, and compute the radius r_c of the orbit. Investigate the stability of this orbit under perturbations that preserve $\mathbf{L}$.