**SMB** Recall that that in a grand canonical ensemble of identical bosons the number of particles in an energy level $\varepsilon_n$ is

$$f(\varepsilon_n) = \frac{1}{e^{\beta(\varepsilon_n - \mu)} - 1}, \quad \mu \leq 0.$$

Consider first free non-relativistic bosons of mass $m$ confined in a large cubic box of volume $V$.

a) Write down, in terms of $\hbar$, $m$ and $V$, the expression $D(\varepsilon)d\varepsilon$ for the number of energy states between $\varepsilon$ and $\varepsilon + d\varepsilon$.

b) The total number of particles $N$ can be written as

$$N = N_0 + N_{\text{excited}} = N_0 + \int_0^\infty D(\varepsilon) f(\varepsilon) d\varepsilon$$

where $N_0$ is the number of particles in the ground state. For a given $k_B T = \beta^{-1}$, at what value of $\mu$ does $N_{\text{excited}}$ take its maximum possible value? Express this maximum value in terms of $\Gamma(s)\zeta(s)$ where these functions are defined below.

c) The critical temperature, $T_c$, is the temperature below which there *must* be a finite fraction $N_0$/N of particles in the ground state. Use your result from part (b) to express $T_c$ as a function of $n = N/V$, the number density of particles.

Now consider the same particles confined in a three dimensional harmonic trap

$$U(x, y, z) = \frac{1}{2} m\omega^2 (x^2 + y^2 + z^2).$$

d) Assuming that the trap is weak (*i.e.* $\hbar\omega \ll k_B T$ so that we can regard the energy spectrum as continuous) write down the new density of states $D(\varepsilon)$.

e) Show that in this case $T_c \propto N^a$ (and not to $n = N/V$) and find $a$.

**Useful Formulæ**:

$$\int_0^\infty \frac{t^{s-1}}{e^t - 1} dt = \Gamma(s)\zeta(s).$$

where $\Gamma(s)$ is Euler's Gamma function and $\zeta(s)$ is the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s > 1.$$