Recall that in a grand canonical ensemble of identical bosons the number of particles in an energy level \( \varepsilon_n \) is

\[
f(\varepsilon_n) = \frac{1}{e^{\beta(\varepsilon_n - \mu)} - 1}, \quad \mu \leq 0.
\]

Consider first free non-relativistic bosons of mass \( m \) confined in a large cubic box of volume \( V \).

a) Write down, in terms of \( \hbar, m \) and \( V \), the expression \( D(\varepsilon)d\varepsilon \) for the number of energy states between \( \varepsilon \) and \( \varepsilon + d\varepsilon \).

b) The total number of particles \( N \) can be written as

\[
N = N_0 + N_{\text{excited}} = N_0 + \int_0^\infty D(\varepsilon)f(\varepsilon)d\varepsilon
\]

where \( N_0 \) is the number of particles in the ground state. For a given \( k_B T = \beta^{-1} \), at what value of \( \mu \) does \( N_{\text{excited}} \) take its maximum possible value? Express this maximum value in terms of \( \Gamma(s)\zeta(s) \) where these functions are defined below.

c) The critical temperature, \( T_c \), is the temperature below which there must be a finite fraction \( N_0/N \) of particles in the ground state. Use your result from part (b) to express \( T_c \) as a function of \( n = N/V \), the number density of particles.

Now consider the same particles confined in a three dimensional harmonic trap

\[
U(x, y, z) = \frac{1}{2}m\omega^2(x^2 + y^2 + z^2).
\]

d) Assuming that the trap is weak (i.e. \( \hbar\omega \ll k_B T \) so that we can regard the energy spectrum as continuous) write down the new density of states \( D(\varepsilon) \).

e) Show that in this case \( T_c \propto N^a \) (and not to \( n = N/V \)) and find \( a \).

Useful Formulae:

\[
\int_0^\infty \frac{t^{s-1}}{e^t - 1} dt = \Gamma(s)\zeta(s).
\]

where \( \Gamma(s) \) is Euler’s Gamma function and \( \zeta(s) \) is the Riemann zeta function

\[
\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}, \quad s > 1.
\]