Center for Academic Resources in Engineering (CARE) Peer Exam Review Session

Math 231 – Calculus II

Mid-Semester Review Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Mar. 23 7:00-9:20pm Greta, Shreya, Sofi, Trusha

Can't make it to a session? Here's our schedule by course:

https://care.grainger.illinois.edu/tutoring/schedule-by-subject

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/844
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1 Improper Integrals and Riemann Sums

1. Determine if the integral converges or diverges:

$$\int_{2\pi}^{\infty} \frac{sin(x) + 4x}{x^2} dx$$

Since $sin(x) + 4x > -1 + 4x$ for all x, we have: $\frac{sin(x)+4x}{x^2} > \frac{-1}{x^2} > \frac{-1}{x^2} + \frac{4}{x} > 0$ for large x. Since $\int_{2\pi}^{\infty} (\frac{-1}{x^2} + \frac{4}{x}) dx$ diverges, so does the given integral by the comparison theorem.

2. Determine if the following integral converges or diverges

$$\int_{1}^{\infty} \frac{x^3 + 12x - 2}{x^6 + 5x^5 + 3x^2 - 1} dx$$

You can use the limit comparison test to prove the behavior of the improper integral

$$f(x) = \frac{x^3 + 12x - 2}{x^6 + 5x^5 + 3x^2 - 1} dx$$

The denominator is bottom heavy by three more powers of x so we expect it to converge

$$g(x) = \frac{1}{x^3}$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^3 + 12x - 2}{x^6 + 5x^5 + 3x^2 - 1} * \frac{x^3}{1}$$

$$\lim_{x\to\infty} \frac{x^6 + 12x^4 - 2x^3}{x^6 + 5x^5 + 3x^2 - 1}$$

$$\lim_{x\to\infty} \frac{x^6 + 12x^4 - 2x^3}{x^6 + 5x^5 + 3x^2 - 1} * \frac{\frac{1}{x^6}}{\frac{1}{x^6}}$$

$$\lim_{x\to\infty} \frac{1 + \frac{12}{x^2} - \frac{2}{x^3}}{1 + \frac{5}{x} + \frac{3}{x^4} - \frac{1}{x^6}} = \frac{1}{1} = 1$$

$\int_1^{\infty} g(x)$ converges by p-test, therefore $\int_1^{\infty} f(x)$ also $\boxed{\text{converges}}$ by Limit Comparison Test

3. Evaluate the following integral using a Trapezoidal Riemann sum with 4 equal intervals:

$$\int_0^4 \frac{x^{0.5}}{2} + 1$$

First we solve for Δx: $\Delta x = \frac{b-a}{n} = \frac{4-0}{4} = 1$

Using the formula for Trapezoidal Riemann sums: $\frac{1}{2}\Delta x(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4))$ where $x_i = a + i\Delta x$

Plugging into the formula:

$T_n = \frac{1}{2}(1)(f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)) = \frac{1}{2}(1 + 2(1.5) + 2(1.7) + 2(1.9) + 2) = 6.6$

2 Partial Fractions and Integration by Parts

4. Evaluate the following integral:

$$\int_1^{e^3} 4x^3 \ln(x)$$

We will need to use integration by parts to solve this problem

$$u = \ln(x) \qquad dv = 4x^3 dx$$

$$du = \frac{1}{x}dx \qquad v = x^4$$

$$\int_1^{e^3} 4x^3 \ln(x)dx = x^4 \ln(x) - \int_1^{e^3} \frac{x^4}{x}dx$$

$$(3e^{12} - 0) - \frac{x^4}{4}\Big|_1^{e^3} = \boxed{\frac{11e^{12} + 1}{4}}$$

5. Evaluate the following integral

$$\int \frac{5x-1}{(x-2)^2(x+1)}$$

First, decompose the given problem using partial fraction decomposition. Recall that there will be two terms for the squared term

$$\frac{5x-1}{(x-2)^2(x+1)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+1}$$

$$5x-1 = A(x-2)(x+1) + B(x+1) + C(x-2)^2$$

$$5x-1 = Ax^2 - Ax - 2A + Bx + B + Cx^2 - 4Cx + 4C$$

$$5x-1 = x^2(A+C) + x(-A+B-4C) + (-2A+B+4C)$$

From this we can get our three equations with three unknowns and solve:

(I) $A+C=0$ (II) $-A+B-4C=5$ (III) $-2A+B+4C=-1$

$9C=-6$

$C=\frac{-2}{3}$

$A=-C=\frac{2}{3}$

$B=5+4C+A=5-\frac{8}{3}+\frac{2}{3}=3$

$$\int \frac{5x-1}{(x-2)^2(x+1)} = \int \frac{\frac{2}{3}}{x-2} + \frac{3}{(x-2)^2} - \frac{\frac{2}{3}}{x+1}dx$$

$$= \boxed{\frac{2}{3}\ln|x-2| + \frac{3}{2-x} - \frac{2}{3}\ln|x+1| + C}$$

6. Evaluate the following integral

$$\int \frac{2x^2 - x + 4}{x^3 + 4x} dx$$

(a) $\ln|x| + \ \ln(x^2+4) - \frac{1}{4}\arctan\left(\frac{x}{2}\right) + C$

(b) $\ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$

(c) $\ln|x| + \frac{1}{2}\ln(x^2+8) - \frac{1}{2}\arctan\left(\frac{x}{4}\right) + C$

(d) $\frac{1}{2}\ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{3}\arctan\left(\frac{x}{2}\right) + C$

(e) $\frac{1}{2}\ln|x| + \frac{1}{2}\ln(x^2+2) - \frac{1}{3}\arctan\left(\frac{x}{4}\right) + C$

$$x^3 + 4x = x(x^2 + 4x)$$

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

$$2x^2 - x + 4 = A(x^2 + 4) + (Bx + C)x$$

$$2x^2 - x + 4 = (A + B)x^2 + Cx + 4A$$

$A + B = 2$, $C = -1$, and $4A = 4$

$A = 1, B = 1, C = -1$

$$\int \frac{2x^2 - x + 4}{x^3 + 4x} dx = \int \frac{1}{x} + \frac{x - 1}{x^2 + 4} dx$$

$$\int \frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4} dx = \boxed{\ln|x| + \frac{1}{2}\ln\left(x^2+4\right) - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C}$$

Note: For the second term, use U-sub where $u = x^2 + 4$. For the third term in the integral, use trig integral for tangent

3 Trig Integrals and Substitutions

7. What is the best substitution to make in order to solve the following integral?

$$\int y^2\sqrt{y^2+4}dy$$

(a) $y = \ln(\theta)$

(b) $y = 2\sec(\theta)$

(c) $y = \sin(\theta)$

(d) $y = 2\tan(\theta)$

(e) $y = e^t + 1$

We want to simplify the radical. Recalling that

$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

the substitution $\boxed{y = 2\tan(\theta)}$ and $dy = 2\sec^2(\theta)$ yields

$$\int 4\tan^2(\theta)\sqrt{4(\tan^2(\theta)) + 4}(2\sec^2(\theta))d\theta$$

$$\int 8\tan^2(\theta)\sec^2(\theta)\sqrt{4(\tan^2(\theta) + 1}d\theta$$

$$\int 16\tan^2(\theta)\sec^3(\theta)d\theta$$

8. Evaluate the following integral

$$\int_0^4 \frac{dx}{(x^2+16)^{\frac{3}{2}}}$$

Note that this is a perfect integral to use TRIG-SUB to solve. Use the following substitution in order to solve the problem:

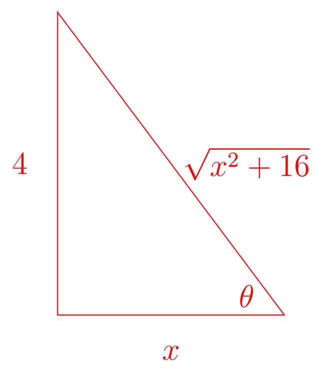
$$x = 4\tan(\theta) \text{ and } dx = 4\sec^2(\theta)$$

$$\int_0^4 \frac{4\sec^2(\theta)}{4^3\sec^3(\theta)}d\theta$$

$$\frac{1}{16}\int_0^4 \frac{1}{\sec(\theta)}d\theta$$

$$\frac{1}{16}\int_0^4 \cos(\theta)d\theta$$

$$\frac{1}{16}\sin(\theta)|_0^4$$



$$\frac{1}{16}\frac{x}{\sqrt{x^2+16}}\Big|_0^4$$

$$\frac{1}{16}\left(\frac{4}{\sqrt{12}}-0\right) = \boxed{\frac{1}{16\sqrt{2}}}$$

9. Evaluate the following Integral:

$$\int_0^{\pi/4} \tan^5(x)\sec^4(x)dx$$

$u = \tan(x)$, $du = \sec^2(x)dx$.

$= \int_0^1 u^5 \sec^2(x)du$

$= \int_0^1 u^5(1+\tan^2(x))du$

$= \int_0^1 u^5(1+u^2)du$

$= \int_0^1 u^5du + \int_0^1 u^7du$

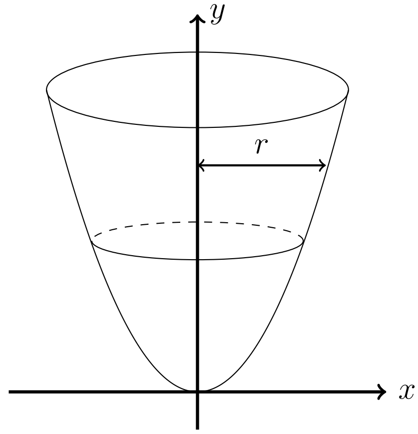
$= \int_0^{\frac{\pi}{4}} \tan^5(x)dx + \int_0^{\frac{\pi}{4}} \tan^7(x)dx$

$= \frac{1}{6}\tan^6(x)|_0^{\frac{\pi}{4}} + \frac{1}{8}\tan^8(x)|_0^{\frac{\pi}{4}}$

$= \frac{1}{6} + \frac{1}{8} = \boxed{\frac{7}{24}}$

4 Physics Applications, Arc Length, Solids of Revolution

10. The figure below represents the calculation of the volume of solid of revolution. Which of the following integrals most closely represents the calculation of the volume as represented in the figure?



(a) $\int_0^1 \pi y^2 dx$

(b) $\int_0^1 \pi x^2 dy$

(c) $\int_0^1 2\pi xy dx$

(d) $\int_0^1 2\pi xy dy$

(e) None of the above

Use the following formula to find volume: $V = \int A(\frac{dy}{dx})$

From the figure, it can be seen that each component of the volume would be a disk of radius x, and they would stack up along the y-direction, therefore: Area $= \pi r^2 = \pi x^2$, with thickness $= dy$.

$$V = \int \pi x^2 dy$$

The bounds of the function are: $y = 0$ to $y = 1$

$$\boxed{V = \int_0^1 \pi x^2 dy}$$

11. Compute the arc length of the function $y = 1 + 2x^{\frac{3}{2}}$ between $x = 0$ and $x = 1$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$y = 1 + 2x^{\frac{3}{2}}, \frac{dy}{dx} = 3x^{\frac{1}{2}}$$

$$S = \int_0^1 \sqrt{1 + (3x^{\frac{1}{2}})^2} dx$$

$$S = \int_0^1 \sqrt{1 + 9x} dx$$

$u = 1 + 9x$, $du = 9dx$, $dx = \frac{1}{9}du$

The u-bounds are: $u = 1 + 9(1) = 10$, $u = 1 + 9(0) = 1$

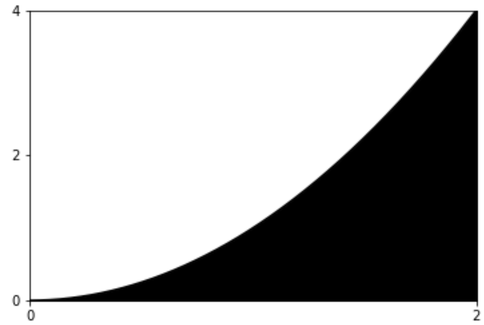
$$S = \int_0^1 \sqrt{u} \frac{1}{9} du$$

$$S = \frac{1}{9}\left(\frac{2}{3}\right) u^{\frac{3}{2}}\Big|_1^{10}$$

$$S = \left(\frac{2}{27}\right)\left(10^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)$$

$$S = \left(\frac{2}{27}\right)(10\sqrt{10} - 1)$$

12. Find the M_x, M_y, and the centroid of $y = x^2$ with density λ on $x \in [0, 2]$.



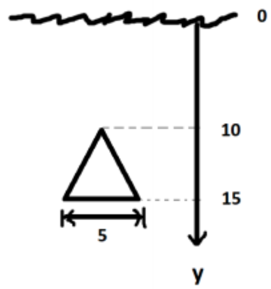
$mass = \lambda \int_0^2 x^2 dx = \frac{\lambda}{3} x^3 |_0^2 = \frac{8\lambda}{3}$

$M_x = \frac{\lambda}{2} \int_0^2 (x^2)^2 dx = \frac{\lambda}{10} x^5 |_0^2 = \boxed{\frac{16\lambda}{5}}$

$M_y = \lambda \int_0^2 x(x^2) dx = \frac{\lambda}{4} x^4 |_0^2 = \boxed{4\lambda}$

Centroid (x,y) $= (\frac{M_y}{mass}, \frac{M_x}{mass}) = \boxed{(\frac{3}{2}, \frac{6}{5})}$

13. Determine the hydrostatic force on the triangle given the density of water $\rho = 1000\text{kg/m}^3$ with a depth y and $g = 9.8\text{m/s}^2$.

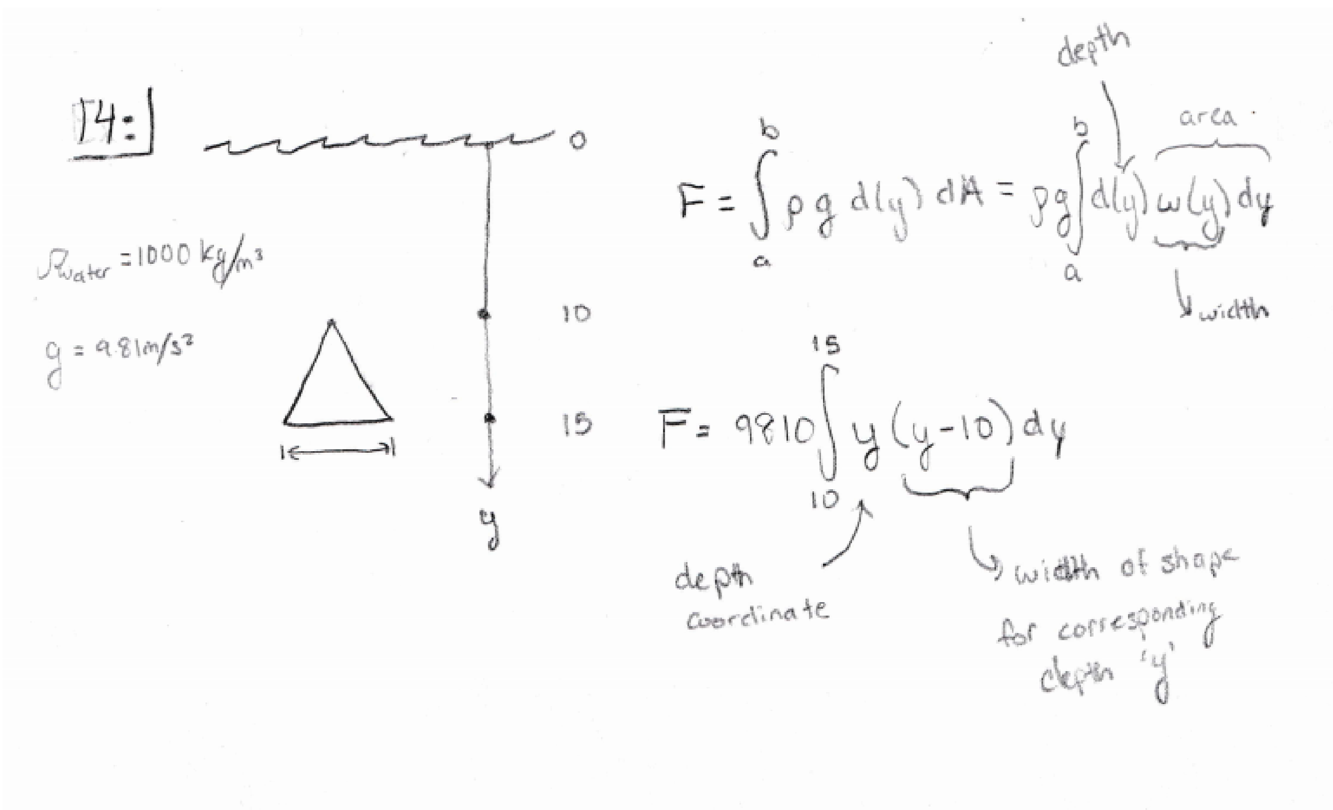


See image below for what each term means within the integral

$F = \int_a^b \rho g d(y) dA = \rho g \int_a^b d(y) w(y) dy$

$F = 9810 \int_{10}^{15} y(y-10) dy$

$\boxed{\text{F} = 1635000N}$



14. Which of the following expressions best represents the volume of the solid of revolution found by rotating the area between the curve $y = 1 + x^2 - 2x^4$ and the x-axis for x on the interval (0,1) around the y-axis?

(a) $2\pi \int_0^1 x\sqrt{1 + (2x - 8x^3)^2}dx$

(b) $2\pi \int_0^1 \sqrt{1 + (2x - 8x^3)^2}dx$

(c) $2\pi \int_0^1 (x + x^3 - 2x^5)dx$

(d) $2\pi \int_0^1 (1 + x^2 - 2x^4)dx$

(e) $2\pi \int_0^1 y\sqrt{1 + (2x - 8x^3)^2}dx$

We want to find the area between the curve $y = 1 + x^2 - 2x^4$ and x-axis for x values in the range of [0,1] around the y-axis

This can be most easily done by using the shell/cylinder method

$$\int_a^b 2\pi xy dx = \int_0^1 2\pi xy dx$$

$$\int_0^1 2\pi x(1 + x^2 - 2x^4)dx$$

$$\boxed{V = 2\pi \int_0^1 (x + 2x^3 - 2x^5)dx}$$