The Grainger College of Engineering
Center for Academic Resources in Engineering

MATH 241

Mid-Semester Review

Keep in mind that this presentation was created by CARE tutors, and while it is thorough, it is not comprehensive.

QR Code to the Queue

The queue contains the worksheet and the solution to this review session

Work Stations

Projector

Dot & Cross Products
Lines & Planes

Station 1

Vector Functions

Station 2

Station 4

Directional Derivatives
Critical Points
Lagrange Multipliers

Station 3

Limits, Chain Rule
Partial Derivatives

Station 1

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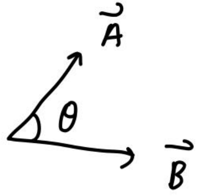
Dot Product

$$\vec{A} = \langle x_1, y_1, z_1 \rangle \qquad \vec{B} = \langle x_2, y_2, z_2 \rangle$$

$$\vec{A} \cdot \vec{B} = x_1 x_2 + y_1 y_2 + z_1 z_2$$

$$= |\vec{A}|\,|\vec{B}| \cos\theta$$



$$\mathrm{proj}_{\vec{A}}\, \vec{B} = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}|^2} \cdot \vec{A}$$

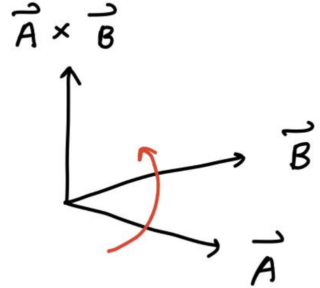
$\vec{B}$ onto $\vec{A}$

$\vec{A} \cdot \vec{B} = 0$, orthoganol vectors

Cross Product

$$\vec{A} = \langle x_1, y_1, z_1 \rangle \qquad \vec{B} = \langle x_2, y_2, z_2 \rangle$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$



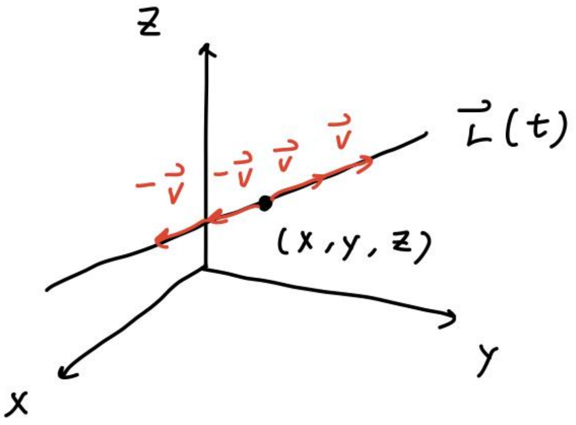
$$= \langle y_1 z_2 - y_2 z_1, \underbrace{x_2 z_1 - x_1 z_2}_{-\hat{j}}, x_1 y_2 - x_2 y_1 \rangle$$

$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin\theta = \text{volume of parallelogram}$$

Equation for Line

$$\vec{L}(t) = \langle x, y, z \rangle + t\,\vec{v}$$

can get this by creating a vector between two points

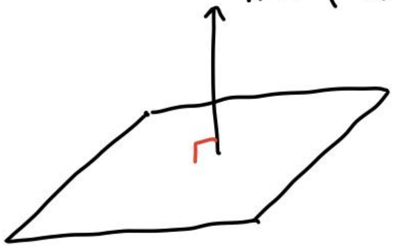


Equation for Plane

$$ax + by + cz + d = 0$$

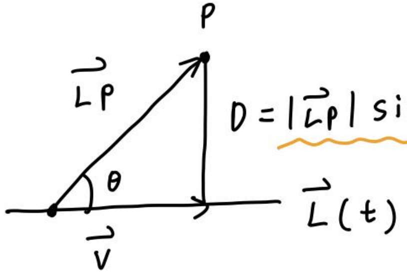
plug in any point on the plane to solve for d

$$\vec{n} = \langle a, b, c \rangle$$



create 2 vectors on the plane and take the cross product

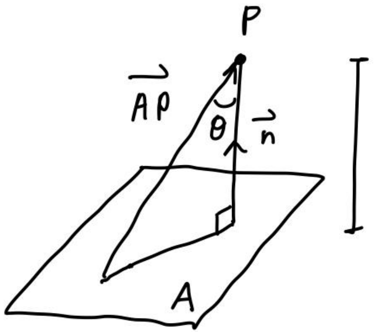
Distances



$$D = |\vec{LP}|\sin\theta$$

$$|\vec{LP}\times\vec{v}| = |\vec{LP}||\vec{v}|\sin\theta$$

$$\rightarrow |\vec{LP}|\sin\theta = \boxed{\frac{|\vec{LP}\times\vec{v}|}{|\vec{v}|} = \text{distance}}$$



$$D = |\vec{AP}|\cos\theta$$

$$\vec{AP}\cdot\vec{n} = |\vec{AP}||\vec{n}|\cos\theta$$

$$\rightarrow |\vec{AP}|\cos\theta = \boxed{\frac{|\vec{AP}\cdot\vec{n}|}{|\vec{n}|} = D}$$

Time to Rotate!

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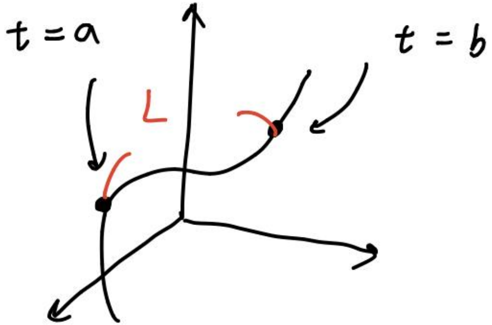
Vector Function

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

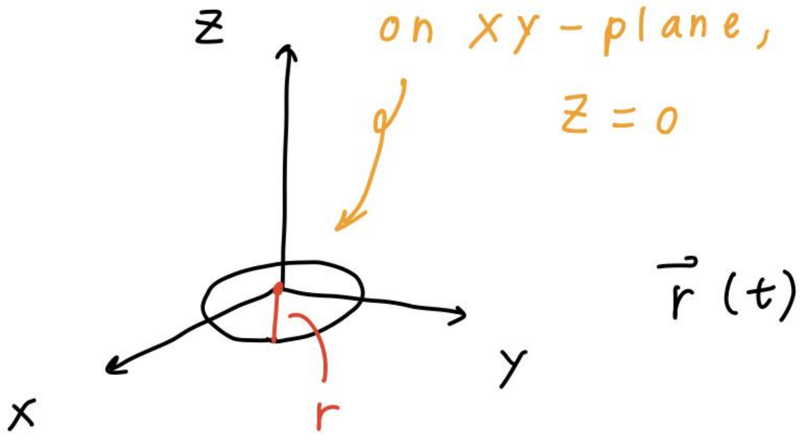
traces a curve in 3D



$$L = \int_a^b |\vec{r}\,'(t)|\, dt$$

$$= \int_a^b \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}\, dt$$

Parameterization of a Circular Curve



$$\vec{r}(t) = \langle r\cos t, r\sin t, 0 \rangle$$

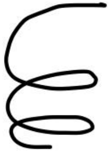
↳ counterclockwise

Identifying Vector Functions

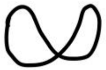
2 sin/cos functions: spirals

$\times t$

t^2



3 sin/cos functions: loops



$\cos(2t)$

determines the number of loops

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How to Solve Limits

(i) plug in (x, y) and check

(ii) check paths to prove DNE

$$x = 0, \quad y = 0, \quad x = y^n, \quad x = -y^n$$

(iii) polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad x^2 + y^2 = r^2$$

$$\lim_{(x,y)\to(0,0)} = \lim_{r\to 0}$$

Partial Derivatives

$$f_x = \frac{\partial f}{\partial x} \rightarrow \text{treat } y \text{ as a constant}$$

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \rightarrow \text{"concavity" in } x\text{-direction}$$

$$f_{xy} = \underbrace{\frac{\partial}{\partial y}}\underbrace{\left(\frac{\partial f}{\partial x}\right)}$$

how does the "slope" in the x-direction changes

as y increases,

Chain Rule

$$f(x(t), y(t))$$

change in f with respect to t

change in x with respect to t

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

change in f with respect to x

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Station 4

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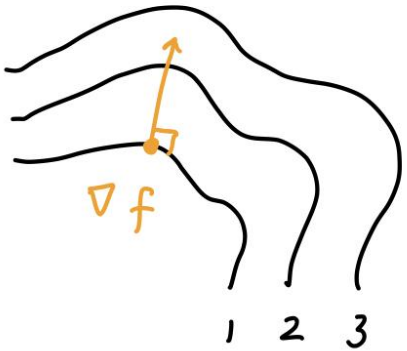
Directional Derivatives

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

$$D_u f(x,y,z) = \nabla f \cdot \vec{u}$$

"slope" in the direction of interest

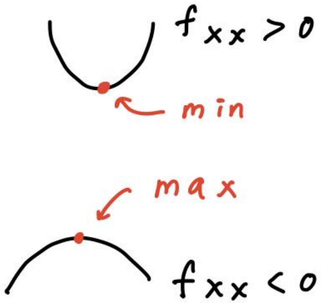


Critical Points & Second Derivative Test

critical point : solve for $f_x = 0$ & $f_y = 0$

$$D = f_{xx} f_{yy} - f_{xy}^2$$

$$\begin{cases} D < 0, \ f_x \text{ or } f_y \neq 0 \rightarrow \text{not critical} \\ D < 0, \ f_x = f_y = 0 \rightarrow \text{saddle} \\ D > 0, \ f_{xx} > 0 \rightarrow \text{local min} \\ D > 0, \ f_{xx} < 0 \rightarrow \text{local max} \end{cases}$$



Lagrange Multipliers

function of interest

constraint (orbits, ...)

$$\nabla f(x, y, z) = \lambda \cdot \nabla g(x, y, z)$$

lagrange multipler

solve for

$$\begin{cases} f_x = \lambda \cdot g_x \\ f_y = \lambda \cdot g_y \\ f_z = \lambda \cdot g_z \\ g(x, y, z) \end{cases}$$

4 unknowns:

x, y, z, λ

Time to Rotate!

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