Math 285

Exam Two Review Session

Disclaimer

- These slides were prepared by tutors that have taken Math 285. We believe that the concepts covered in these slides could be covered in your exam.
- HOWEVER, these slides are NOT comprehensive and may not include all topics covered in your exam. These slides should not be the only material you study.
- While the slides cover general steps and procedures for how to solve certain types of problems, there will be exceptions to these steps. Use the steps as a general guide for how to start a problem but they may not work in all cases.

Second Order ODEs

- Standard Form: $y'' + p(t)y' + q(t)y = g(t)$
- Homogeneous Equations occur when $g(t) = 0$.
- Theorem of Superposition:
  - If y_1, y_2 are both solutions of a linear homogeneous equation, then the linear combination $y = c_1 y_1 + c_2 y_2$ is also a solution for any c_1, c_2 in all real numbers.

2nd Order ODEs with Constant Coefficients

- For ODEs of the form: ay” + by’ + cy = 0 where a, b, and c are all constants, we can write a characteristic equation to solve the ODE.
- The characteristic eqn will be of the form a(r^2) + br + c = 0.
- From there, we can factor this equation like a quadratic and solve for r.
- Once we have r, we can write a general solution for the ODE.
- General Solution: y(t) = c1*e^(r1*t) + c2*e^(r2*t)
- Finally consider any initial conditions

$$3y'' + 2y' - 8y = 0$$

2nd Order ODEs with Complex Roots

- The process for solving these equations is the same as the process for solving 2nd order equations with constant coefficients.
- The difference is the format of the roots from the characteristic equation.
- Roots will take the form of $a + bi$.
- The general solution formula then becomes:
  - y(t) = e^(at) * [c1*cos(bt) + c2*sin(bt)]

2nd Order ODEs with Repeated Roots (Roots of Higher Multiplicity)

- The process for solving these equations is the same as the process for solving 2nd order equations with constant coefficients.
- The difference is the format of the roots from the characteristic equation.
- Roots will take the form of just r.
- The general solution formula then becomes:
  - y(t) = c1*e^(rt) + c2*t*e^(rt)

Summary of Types of 2nd Order ODE Solutions

- Given a 2nd order ODE of the form y” + p(t)*y’ + q(t)*y = g(t) where p(t) and q(t) are constants and g(t) = 0, the following root types can occur from the homogeneous or characteristic solution:
    - **Unique Roots:** when r1 does not equal r2
        - General Solution: y(t) = c1*e^(r1*t) + c2*e^(r2*t)
    - **Complex Roots:** when r = a + bi
        - General Solution: y(t) = e^(a*t)*[c1*cos(b*t) + c2*sin(b*t)]
    - **Repeated Roots:** when r1 equals r2
        - General Solution: y(t) = c1*e^(r*t) + c2*t*e^(r*t)

Method of Undetermined Coefficients

- Used to solve non-homogeneous ODEs that have a right hand side with an exponential, polynomial, or sin/cos term.
  - Ex: y” + p(t)*y’ + q(t)*y = g(t) when g(t) = a constant or function
- General solution will be of the form y(t) = Yp + Yc, where Yp is the particular solution and Yc is the characteristic solution.
- First, find the characteristic solution.
- Second, find the particular solution expression based on the right side of the equation
- Third, plug particular solution back in to solve for coefficients.

Common Particular Solutions and Example

Common Particular Solution Expressions	
$f(t)$	$y_p(t)$
kt^m	$A + Bt + \cdots Ct^m$
ke^{at}	Ae^{at}
$ksin(\omega t)$	$Acos(\omega t) + Bsin(\omega t)$
$kcos(\omega t)$	$Acos(\omega t) + Bsin(\omega t)$
$k * e^{at} * sin(\omega t)$	$e^{at}(Acos(\omega t) + Bsin(\omega t))$
$k * e^{at} * cos(\omega t)$	$e^{at}(Acos(\omega t) + Bsin(\omega t))$

- Important Note: Find the homogeneous/characteristic solutions first. Then when you pick your particular solution expression, you know if you need to multiply a term by t to ensure linear independence.

Annihilator Method

- First, identify the corresponding annihilators for each term on the right hand side of your ODE.
- Second, multiply the whole ODE by the annihilator to get rid of the right hand side terms.
- Lastly, solve the resulting homogeneous ODE with a characteristic equation solution.

$f(t)$	Annihilator **M**
1	$\frac{d}{dt}$
$P_n(t)$	$\frac{d^{n+1}}{dt^{n+1}}$
e^{at}	$\frac{d}{dt} - a$
$P_n(t)e^{at}$	$(\frac{d}{dt} - a)^{n+1}$
$A\cos(bt) + B\sin(bt)$	$\frac{d^2}{dt^2} + b^2$
$P_n(t)\cos(bt) + Q_n(t)\sin(bt)$	$(\frac{d^2}{dt^2} + b^2)^{n+1}$
$A\cos(bt)e^{at} + B\sin(bt)e^{at}$	$(\frac{d}{dt} - a)^2 + b^2$
$P_n(t)\cos(bt)e^{at} + Q_n(t)\sin(bt)e^{at}$	$((\frac{d}{dt} - a)^2 + b^2)^{n+1}$

Oscillators

- General expression: $mu'' + \gamma u' + ku = F(t)$
  - m = mass
  - γ = frictional coefficient or damping factor
  - k = spring constant
- When F(t) = 0, the oscillation is free and has no external forces.
- An oscillator is undamped when $\gamma = 0$.
- Natural frequency (w0) = sqrt(k/m)
- Period of an oscillation can be described as T = (2*pi)/(w0)
- If F(t) = Acos(wt) or Asin(wt) and w = w0, the system is experiencing resonance.

Types of Oscillators

- Overdamping sqrt(γ^2 - 4*m*k) > 0
  - Characteristic solution will have 2 distinct real roots
  - u(t) = A*e^(r1*t) + B*e^(r2*t)
- Critically Damped sqrt(γ^2 - 4*m*k) = 0
  - Characteristic solution will have repeated roots
  - u(t) = (At + B)*e^-(γ/2*m)*t
- Underdamping sqrt(γ^2 - 4*m*k) < 0
  - Characteristic solution will have complex roots
  - u(t) = (e^-(γ/2*m)*t)*(A*cos(wt) + B*sin(wt))

RLC Oscillators

- 2nd order ODEs can also be useful for modeling the current or voltage in RLC circuits.
- Similarities can be seen between Mechanical and Electrical models.

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dV}{dt}$$

Mechanical	Electrical
Mass M (inertia)	Inductance
Damping γ	Resistance R
Restoring force k	Inverse capacitance $\frac{1}{C}$
External Force $F(t)$	Rate of change of voltage $\frac{dV}{dt}$
Displacement $y(t)$	Current $I(t)$