Math 285

Exam One Review Session
Disclaimer

- These slides were prepared by tutors that have taken Math 285. We believe that the concepts covered in these slides could be covered in your exam.
- HOWEVER, these slides are NOT comprehensive and may not include all topics covered in your exam. These slides should not be the only material you study.
- While the slides cover general steps and procedures for how to solve certain types of problems, there will be exceptions to these steps. Use the steps as a general guide for how to start a problem but they may not work in all cases.
Differential Equations

- “A differential equation is an equation consisting of functions and their derivatives.” Simply, the rate of change.
- **Slope Fields**: Help visually model the Differential Equation
  - The solutions to a DE are tangent to the direction field.
  - Values with a slope of zero are the equilibrium solutions or the long term behavior.

Population dynamics: \( \frac{dp}{dt} = kp - r \ (k, r > 0) \)
Classifying Differential Equations

- **Ordinary Differential Equations (ODEs):** The unknown function depends on just one independent variable.
  - Ex. \( \frac{dy}{dx} = y + x \) or \( y' = y - kx \)

- **Partial Differential Equations (PDEs):** The unknown function depends on two or more independent variables.
  - Ex. Heat equation: \( u_t = k u_{xx} \) or Wave Equation: \( u_{tt} = c u_{xx} \)
Classifying Differential Equations Cont.

- **Order of a Differential Equation:** The magnitude of the highest derivative in the equation.
- **Linear/Nonlinear Differential Equations:** The coefficients of your unknown function can’t be derivatives of itself.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Linear or not</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p' = kp - r$</td>
<td>linear</td>
</tr>
<tr>
<td>$yy' = t^2$</td>
<td>nonlinear (due to $yy'$)</td>
</tr>
<tr>
<td>$y'' + t^3y = \cos t$</td>
<td>linear</td>
</tr>
<tr>
<td>$y'' + (t - 1)y = \sin y$</td>
<td>nonlinear (due to $\sin y$)</td>
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Existence and Uniqueness Theorem

- Consider the initial value problem: $y' + p(t)*y = q(t)$, $y(t_0) = y_0$.
- If $p$ and $q$ are continuous on an open interval $(a,b)$, with $t_0$ on $(a,b)$, then our initial value problem has a unique solution on $(a,b)$.
- Essentially, we need to see if the equation has any discontinuities or asymptotes that we need to consider.

Consider the initial value problem $ty' + 2y = \frac{3}{t}$, $y(1) = 2$.

Answer: standard form is $y' + \frac{2}{t}y = \frac{3}{t^2}$. We have $p(t) = \frac{2}{t}$, $q(t) = \frac{3}{t^2}$. These functions are continuous on the open interval $(0, \infty)$, which contains the initial point $t = 1$, hence our initial value problem has a unique solution on $(0, \infty)$. 
Solving ODEs: Separation of Variables

General Procedure:

1. Put all derivatives and variables of the same form on the same side.
2. Integrate both sides
3. Simplify and solve for the unknown function that solves the ODE.
Solving ODEs: Integrating Factors

Step 1: Put ODE in standard form. $y' + p(t)y = q(t)$

Step 2: Take the integral of $p(t)$. Don’t include $+C$.

Step 3: Compute the integrating factor, $\mu(t) = e^{\int p(t) \, dt}$.

Step 4: Multiply ODE by $\mu(t)$. This puts the ODE in the form: $\mu*y' + y*\mu' = \mu*q(t)$.

Step 5: Apply Product Rule to the left side. $[\mu*y]' = \mu(t)*q(t)$.

Step 6: Integrate both sides and solve for $y(t)$. 
Exact Equations

- Identifying first order exact equations can be another method for solving ODEs.

**Definition**

An equation of the form

$$N(x, y) \frac{dy}{dt} + M(x, y) = 0$$

is exact if the functions $N(x, y)$, $M(x, y)$ satisfy the condition

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$
Solving Exact Equations

If we can find $\psi(x, y)$ we can solve the equation. The following algorithm lets us find $\psi$.

**Theorem**

*Give $N(x, y), M(x, y)$ such that $N_x = M_y$ we can find a function $\psi(x, y)$ such that $\psi_y = N(x, y), \psi_x = M(x, y)$*

1. Take one of the equations, say $\psi_x = M(x, y)$
2. Take the "partial integral" with respect to $x$ (treating $y$ as constant).
3. The "constant" of integration will be a function of $y$.
4. Take the derivative of the resulting $\psi$ with respect to the remaining variable (in this case $y$)
5. Use the other condition to determine the unknown function.
Some ODEs are not explicitly dependent on their input (x or t) and can have the following form: \(\frac{dy}{dx} = (y + a)(y^2 - b)\).

An equilibrium point or solution is when \(\frac{dy}{dx} = 0\). In other words, values of \(y\) that make the equation zero.

Ex. \(\frac{dy}{dx} = (y + a)(y^2 - b)\)

- Equilibrium points: \(\frac{dy}{dx} = 0\) when \(y = -a, b,\) and \(-b\)

Phase lines can then be used to help visualize how potential solutions behave between these points.
Existence/Uniqueness and Linear Dependence

- **Theorem of Existence and Uniqueness**: If the functions p, q, and g are continuous on an interval that contains t_0, then the initial value problem y'' + p(t)y' + q(t)y = g(t) with y(t_0) = y_0, y'(t_0) = y'_0 has a unique solution on that interval.

- Two functions, N(t) and M(t), are **linearly dependent** if one is a real multiple of the other, meaning N(t) = c*M(t) or c*N(t) = M(t).

- Two functions are linearly independent if they are **NOT** multiples of each other.

- The Wronskian can be used to determine if two solutions are independent. 
  - **NOTE**: If the Wronskian is equal to zero, this does not prove linear dependence.