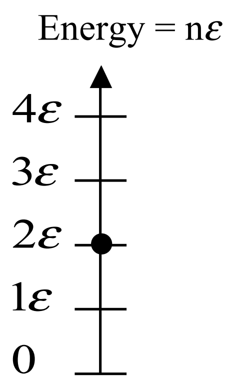
Solution

Physics 213 **Problem 3** **Week 4**

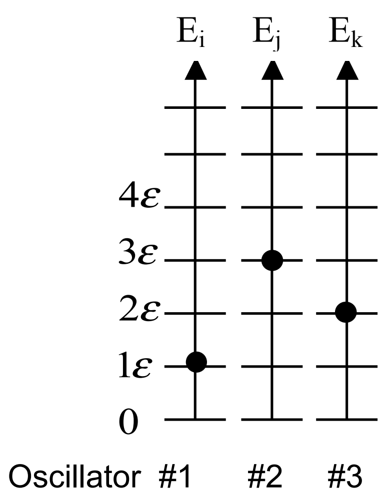
Energy Microstates

Thermodynamic processes almost always involve the exchange of energy between subsystems. To begin our study of energy exchange we consider the simplest possible system: a **harmonic oscillator** such as a vibrating diatomic molecule. The molecule can absorb or emit only an integral number of energy packets. We write the energy levels of an oscillator as a ladder of equally spaced energy levels, where epsilon (ε) is the quantum of energy:



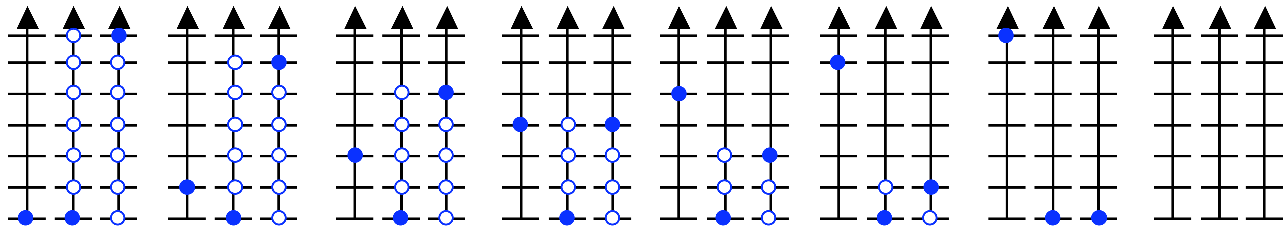
<u>Convention</u>:
E = single-particle energy
U = energy of a many-particle system

The oscillator shown has an energy of 2ε; that is, it possesses two quanta (or packets) of energy. Now imagine three such oscillators which are able to exchange energy. One possible **microstate** of the 3-particle system is the following:



a) What is the total energy U of the situation depicted above, in terms of ε? U = __6ε__. In general we will write U = qε, where q is the total number of energy quanta in the system. From the convention above, U represents the total energy of all of the oscillators. So all we need to do is add the number of quanta that are in each of the oscillators. Thus **U = 6ε.**

b) Determine the number of microstates for the three-oscillator system with a total energy of $U = 6\varepsilon$.
[Hint: Start with $E_i = 0$ and count the # of ways you could arrange 6ε in oscillators 2 and 3. Next set $E_i = \varepsilon$ and count the # of ways you could arrange 5ε in oscillators 2 and 3. Continue, and add up all the possibilities. Ω = *total* # microstates. Here is some workspace:



Each diagram shows all the combinations for one choice of E_i. For example, when $E_i = 0$, E_j can take any value between 0 and 6 (i.e., 7 values). For each value of E_j , E_k is constrained by energy conservation. So, there are a total of 7+6+5+4+3+2+1 = 28 total microstates for $U = 6\varepsilon$.

c) Now fill out the table below by counting the number of microstates Ωq for the three-oscillator system for the total energies shown. To save time, notice how the 6ε answer is changed when you go to 5ε, etc. Check your answer with the formula given below, a derivation of which will be given in the posted solutions.

$U = q\varepsilon$	Ω_q
0	1
1ε	3
2ε	6
3ε	10
4ε	15
5ε	21
6ε	28

$$\Omega_q = \frac{(q + N - 1)!}{(N-1)!\,q!}$$

q = # quanta N = # oscillators

The calculation for each q is similar to the q = 6 case.

This equation can be derived from the binomial coefficient, $C_b(N,n)$.
We want to distribute q quanta into N bins. Putting the N bins in a row, we see that there are N+1 walls. So, a typical microstate will look like this: (for q = 7and N = 9)

| * | | * * | * | | | | * * | * |

where we've denoted energy quanta by * and walls by |.
Two adjacent * indicates multiple occupancy. Two adjacent | indicates an empty bin. In the diagram, the occupancies of the 9 bins are: (1,0,2,1,0,0,0,2,1)

Note that the diagram must begin and end with |, because all energy must be in bins. We want to count the number of permutations of the N-1 remaining walls and the q quanta (*i.e.*, q+N-1 objects), remembering that the walls are indistinguishable, as are the quanta. Thus, the answer is:

$$\Omega_q = \frac{(q+N-1)!}{(N-1)!\; q!}$$