Solution

Physics 213 **Problem 4** **Week 5**

Oscillators, Temperature and Equipartition

Consider a collection of ONE MILLION oscillators, each with energy-level spacing $\varepsilon = 10^{-23}$ J. Let there be 10^{-14} J of total thermal energy in this system.

a) On the average, how many energy quanta does each oscillator get?
The number of quanta is q = (total energy / energy per quantum)
$= U/\varepsilon = 10^{-14}$ J / 10^{-23} J $= 10^{9}$.
The average number of quanta per oscillator is $q / N = 10^{9} / 10^{6} =$ **1000**.

b) Is the collection near the $q >> N >> 1$ limit, in which classical equipartition applies? Below approximately what temperature would we need to start accounting for the fact that the oscillator degrees of freedom are "frozen out"?
$10^{9} >> 10^{6} >> 1$, so **yes**.
The point here is that we can use the equipartition theorem when $q >> N >> 1$.
When *won't* this be true? When $kT < \varepsilon = 10^{-23}$ J, or when $T < \varepsilon/k \sim 1$ K.

c) What is the heat capacity C of this system? [HINT: Use equipartition and remember that a 1-dimensional oscillator has two quadratic modes.]
Note that for oscillators in 1 dimension, there are two quadratic degrees of freedom, one for PE and one for KE. Then since the equipartition theorem holds, $U = NkT = C_V T$. Note that for oscillators, there is only one heat capacity, $C = C_V$, since the volume does not change.
$C = C_V = Nk = 10^{6} \times 1.38 \times 10^{-23}$ J/K $=$ $\mathbf{1.38 \times 10^{-17}}$ **J/K**

d) What is the temperature, T?
Because the heat capacity is constant, $U = CT$.
$T = U / C = 10^{-14}$ J / (1.38×10^{-17} J/K) = **725 K**

e) What is the average energy of one of the oscillators?
$\langle E \rangle = \langle q \rangle \varepsilon = (1000)(10^{-23}\text{ J}) = 10^{-20}$ J

f) The probability that an oscillator has energy E is proportional to the Boltzmann factor, exp(-E/kT). We can calculate the normalization factor ("Z") if we treat energy as a continuous variable (OK when $q >> N$). Then we can do an integral instead of a sum. Calculate the normalized probability density, P(E), by integrating the Boltzmann factor over all possible energies and requiring that the total probability = 1.

You need to integrate from 0 to infinity: $\int_0^\infty e^{-E/kT} dE = kT$. So, $P(E) = \frac{1}{kT} e^{-E/kT}$.

$$< E > \; = kT = 10^{-20}\,\mathrm{J} \qquad \Delta E = 0.1 < E > \; = 10^{-21}\,\mathrm{J} \qquad (\pm 5\%\ \textit{of average energy})$$

$$P(E)\,\Delta E = \frac{1}{kT} e^{-E/kT} \Delta E \; = \frac{10^{-21}\,\mathrm{J}}{10^{-20}\,\mathrm{J}} e^{-10^{-20}\,\mathrm{J}/10^{-20}\,\mathrm{J}} \; = \frac{e^{-1}}{10} \; = \; 0.037 \quad \textit{near} \quad E = < E >$$