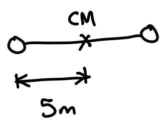
Rotating Astronauts

Two astronauts, each having a mass 75 kg, are connected by a light rope 10 meters long. They are isolated in space, and are initially revolving around their common center of mass once every 6 seconds. They now pull in on the rope, halving the distance between them. How much work is done by the astronauts as they shorten the rope?

Since this is a physics class, we'll assume the astronauts are point masses.



Thus,

$$I_i = MR_i^2 + MR_i^2 = 2MR_i^2$$

Then they pull in halfway so that $R_f = \frac{R_i}{2}$

$$I_f = M\left(\frac{R_i}{2}\right)^2 + M\left(\frac{R_i}{2}\right)^2 = \frac{1}{2}MR_i^2$$

When they pull on the rope, angular momentum is conserved:

$$\vec{L}_i = \vec{L}_f = \vec{L}$$

where $\vec{L}_i = I_i\vec{\omega}_i$

and $\omega_i = \frac{2\pi}{T} = \frac{\pi}{3}\ \text{rad/s}$

period (not tension)

So we have

$$\vec{L} = (2MR_i^2)\left(\frac{\pi}{3}\right) = \frac{2\pi}{3}MR_i^2$$

We can use the work-energy theorem to find the total work done:

$$W = \Delta K \qquad \text{where } K = \frac{L^2}{2I}$$

$$= \frac{L^2}{2I_f} - \frac{L^2}{2I_i}$$

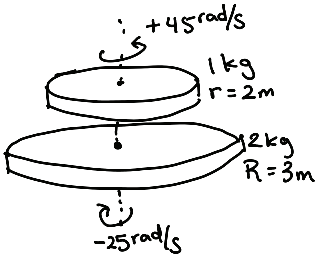
Plug in the values for L, I_i, and I_f to find

$$\boxed{W = 6168.5\,\text{J}}$$

Spinning Disks

Two disks are mounted on frictionless bearings on the same axle and can be brought together so that they couple and rotate as one unit. The first disk, which has a mass of 1 kg and radius of 2 m, is set spinning at 45 rad/s. The second disk, which has a mass of 2 kg and a radius of 3 m, is set spinning at 25 rad/s in the opposite direction. They then couple together. What is their angular speed after coupling?

Before:



When the disks couple, they experience no net external torque, so angular momentum is conserved.

Top disk:

$I_{top} = \frac{1}{2} m r^2 = 2\,kg\,m^2$

$\omega_{top} = +45$ rad/s

$L_{top} = I_{top}\,\omega_{top} = 90\,kg\,m^2/s$

Bottom disk:

$I_{bottom} = \frac{1}{2} M R^2 = 9\,kg\,m^2$

$\omega_{bottom} = -25$ rad/s

$L_{bottom} = I_{bottom}\,\omega_{bottom} = -225\,kg\,m^2/s$

Angular momentum is conserved, so

$L_{top} + L_{bottom} = L_{coupled} = -135\,kg\,m^2/s$

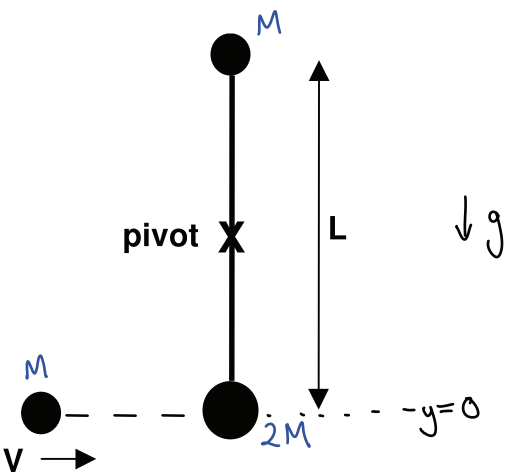
and $L_{coupled} = (I_{top} + I_{bottom})\omega$

Plug in I_{top}, I_{bottom}, and $L_{coupled}$ to find

$$\boxed{\omega = 12.27\ \text{rad/s}}$$

Dumbbell Collision

A dumbbell consists of two balls, one of mass M and the other of mass 2M, connected by a light rod of length L. The dumbbell is mounted vertically on a pivot with the heavier ball at the bottom. The pivot is located at the midpoint of the rod. The system, which is initially at rest, is free to rotate about the pivot. A wad of putty of mass M and initial velocity V collides with and sticks to the lower mass, as shown in the diagram. In terms of the quantities given, what is the minimum value of V for which the dumbbell will make it all the way around?



When the putty collides with the lower mass, $\tau_{ext} = 0$, so $\vec{L}$ is conserved:

Before the collision:

$$\begin{cases} I_{putty} = MR^2 = M\left(\frac{L}{2}\right)^2 = \frac{1}{4}ML^2 \\ \omega_{putty} = \frac{V}{R} = \frac{2V}{L} \\ L_{putty} = I_{putty}\,\omega_{putty} = \frac{MLV}{2} \end{cases}$$

immediately after the collision:

$$\begin{cases} I_{total} = I_{putty} + I_{top} + I_{bottom} = \frac{1}{4}ML^2 + \underbrace{M\left(\frac{L}{2}\right)^2}_{I_{top} = MR^2} + \underbrace{2M\left(\frac{L}{2}\right)^2}_{I_{bottom} = M_{bottom}R^2} \\ \quad = ML^2 \\ L_{total} = L_{putty} = \frac{MLV}{2} \end{cases}$$

As the dumbbell swings around, mechanical energy is conserved:

at the bottom $\rightarrow$ $E_i = K_i + U_i$ where $K_i = \frac{L_{total}^2}{2I_{total}} = \frac{MV^2}{8}$ and $U_i = MgL$ (recall the top mass is at height $y = L$)

$$\boxed{E_i = \frac{MV^2}{8} + MgL}$$

$E_f = K_f + U_f$ where $K_f = 0$ and $U_f = 3MgL$ (the putty and larger end of the dumbbell are at height $y = L$)

$$\boxed{E_f = 3MgL}$$

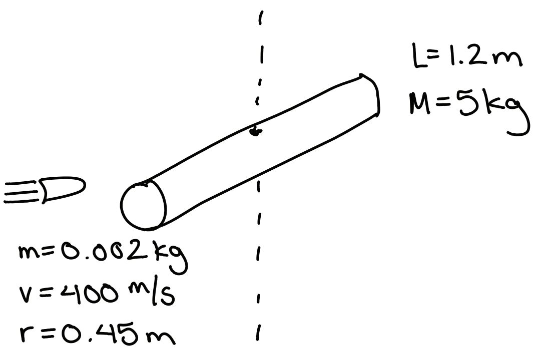
Set $E_i = E_f$ to find

$$\boxed{V = \sqrt{16gL}}$$

Faster than a Speeding Bullet

You are assigned the job of designing a simple system to measure the speed of a bullet. The system you come up with works in the following way: You shoot the bullet into a hardwood rod of mass of 5 kg and a length of 1.2 m, mounted on a frictionless axle which passes through it's center and is perpendicular to its length. The axle is vertical, so the rod rotates in the horizontal plane after the bullet hits it. The rod is initially at rest and oriented perpendicular to the path of the bullet. You shoot a bullet of mass 2.0 grams at 400 m/s into the rod a distance of 0.45 m to one side of the axle, where it embeds itself and stops.

1. How much time will it take the rod to make one full revolution after the bullet slams into it?
2. What is the ratio of the initial to the final kinetic energy of the system?



when the bullet hits the rod, $\vec{L}$ is conserved. Thus, we begin by finding $\vec{L}_{bullet}$.

$$I_{bullet} = mr^2 = 4.05 \times 10^{-4} \text{ kg m}^2$$

$$\omega_{bullet} = \frac{v}{r} = 888.89 \text{ rad/s}$$

$$\vec{L}_{bullet} = I_{bullet}\,\omega_{bullet} = 0.36 \text{ kg m}^2/\text{s}$$

1. We find the time of one rotation by finding ω_{final} and using kinematics:

$$\vec{L}_{bullet} = \vec{L}_{final} = I_{final}\,\omega_{final}$$

where $I_{final} = I_{rod,CM} + I_{bullet} = \frac{1}{12}ML^2 + (4.05 \times 10^{-4})$
$= 0.600405 \text{ kg m}^2$

then $\omega_{final} = \dfrac{L_{final}}{I_{final}} = 0.6 \text{ rad/s}$

when $\alpha = 0$, we know $\theta = \omega t$, so $t = \dfrac{\theta}{\omega} = \dfrac{2\pi}{0.6} = \boxed{10.48\text{s}}$

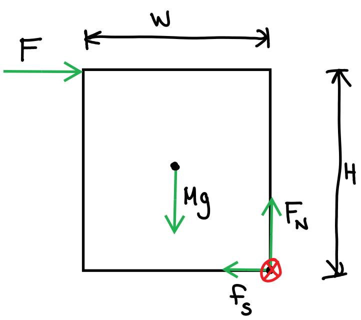
2. The easiest way to find the ratio K_i/K_f is with $K = \dfrac{L^2}{2I}$:

$$\frac{K_i}{K_f} = \frac{\cancel{L^2}/\cancel{2}I_i}{\cancel{L^2}/\cancel{2}I_f} = \frac{I_f}{I_i} = \frac{I_{final}}{I_{bullet}} = \boxed{1482.5}$$

Review Rotational Statics: Tipping Crates

A uniform rectangular crate of height H and width W weighs Mg. It is desired to tip (i.e., roll) the crate by applying a horizontal force F to one of the upper edges.

(a) Assuming that the crate does not slip, what is the minimum force necessary to tip the crate?

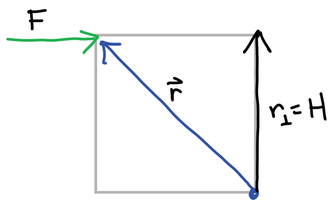


Rotate about tipping point

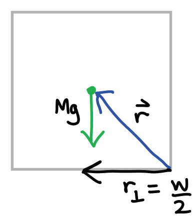
Just as the crate tips, the static friction force and normal force move to the remaining contact point, as shown.

$\sum \tau = 0$ where $\sum \tau = \tau_F + \tau_{Mg} + \cancel{\tau_{f_s}}^{\,0} + \cancel{\tau_{F_N}}^{\,0}$

$$\vec{\tau}_F = \vec{r}_F \times \vec{F} = -r_\perp F = -HF$$



$$\vec{\tau}_{Mg} = \vec{r}_{mg} \times M\vec{g} = r_\perp Mg = \left(\frac{W}{2}\right)Mg$$



Thus we have

$$\frac{MgW}{2} - FH = 0$$

$$\boxed{F = \frac{MgW}{2H}}$$

(b) What minimum coefficient of static friction between the crate and the ground is required such that the crate tips but does not slide?

We call in our linear dynamics equations for this part:

$\sum F_x = 0$:

$$F - f_s = 0 \rightarrow f_s = F = \frac{MgW}{2H}$$

$\sum F_y = 0$:

$$F_N - Mg = 0 \rightarrow F_N = Mg$$

To have a minimum μ_s, we maximize f_s:

$$f_{s,max} = \mu_s F_N$$

With the values found above, we then have

$$\frac{MgW}{2H} = \mu_s Mg$$

$$\boxed{\mu_s = \frac{W}{2H}}$$