The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Mar 4, 2-4pm Geitanksha and Marco
Session 2: Mar 5, 12-2pm Alberto, Mukhil, and Trusha

Can’t make it to a session? Here’s our schedule by course:

https://care.grainger.illinois.edu/tutoring/schedule-by-subject

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/844
2. Click “New Question”
3. Add your NetID and Name
4. Press “Add to Queue”

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!
1. Determine whether the series converges or diverges:

$$\sum_{n=5}^{\infty} \frac{1}{n(\ln(n))^2}$$

Use the integral test.

$$= \int_{5}^{\infty} \frac{1}{x(\ln(x))^2} dx \quad [u = \ln(x), \ du = \frac{1}{x}]$$

$$= -\frac{1}{\ln(x)} \bigg|_{5}^{\infty} = \frac{1}{\ln(5)} \rightarrow \text{so the integral converges. Both converge.}$$

2. Determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{n^2}{n^2 + 1}$$

Use the divergence test:

$$\lim_{n \to \infty} \frac{(n^2)}{n^2 + 1} = 1 \rightarrow \text{series diverges.}$$

3. The sum $$\sum_{n=1}^{\infty} \frac{1}{n^3}$$ converges, yet has no known closed-form expression (i.e. We don’t know the exact value of the sum). Use the Integral Test to find the smallest and largest possible values for the sum.

The integral test is valid here, since $$f(n) = \frac{1}{n^3}$$ is a positive, decreasing function that goes to 0. We may conclude that

$$\int_{1}^{\infty} \frac{1}{x^3} \ dx \leq \sum_{n=1}^{\infty} \frac{1}{n^3} \leq \frac{1}{1^3} + \int_{1}^{\infty} \frac{1}{x^3} \ dx$$

$$\frac{1}{2} \leq \sum_{n=1}^{\infty} \frac{1}{n^3} \leq \frac{3}{2}$$

In fact, $$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.2020.$$

4. Determine whether the series converges or diverges:

$$\sum_{n=3}^{\infty} \frac{1}{n \ln(n)}$$

Use the integral test:
\[ f(x) = \frac{1}{x \ln(x)} \] is positive and decreasing and continuous between 3 and \( \infty \).

\[
\int_3^\infty \frac{1}{x \ln(x)} = \ln(\ln(x))\bigg|_3^\infty = \ln(\ln(\infty)) - \ln(\ln(3)) = \infty
\]

The integral diverges, so the series diverges as well.

5. Find the values of \( x \) for which the following series converges, and then find the sum of the series for those values:

\[
\sum_{n=0}^\infty \frac{\cos^n(x)}{3^n}
\]

Identify the geometric series:

\[
\sum_{n=1}^\infty ar^{n-1} \quad \text{where} \quad r = \frac{(n+1)-\text{term}}{(n)-\text{term}} = \frac{\cos^{n+1}(x)}{3^{n+1}} \cdot \frac{3^n}{\cos^n(x)} = \frac{\cos(x)}{3}
\]

\[
= \frac{\cos(x)}{3} = \frac{1}{3} \cos(x), \text{ which is always } < 1, \text{ so the series converges for all } x.
\]

Converges to \( \frac{a}{1-r} \rightarrow a = 1 \) and \( r = \frac{\cos(x)}{3} \rightarrow \frac{1}{1-\cos(x)} = \frac{3}{3-\cos(x)} \)

6. Determine whether the series converges. If it converges, find what it converges to.

\[
\sum_{n=1}^\infty \frac{1}{(\frac{1}{2})^n}
\]

\[
r = \frac{(n+1)-\text{term}}{(n)-\text{term}} = \frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}} = \frac{1}{2} = 2 \rightarrow \text{diverges.}
\]

7. Determine whether the series converges. If it converges, find what it converges to.

\[
\sum_{n=1}^\infty \frac{1}{2^n}
\]

\[
r = \frac{(n+1)-\text{term}}{(n)-\text{term}} = \frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}} = \frac{1}{2} \rightarrow \text{converges.}
\]

\[
\frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{1}{2} = 1
\]
8. Find the $M_x$, $M_y$, and the centroid of $y = x^2$ with density $\lambda$ on $x \in [0, 2]$.

$$\text{mass} = \lambda \int_0^2 x^2 \, dx = \frac{\lambda}{3} x^3 \bigg|_0^2 = \frac{8\lambda}{3}$$

$$M_x = \frac{\lambda}{2} \int_0^2 (x^2)^2 \, dx = \frac{\lambda}{10} x^5 \bigg|_0^2 = \frac{16\lambda}{5}$$

$$M_y = \lambda \int_0^2 x(x^2) \, dx = \frac{\lambda}{4} x^4 \bigg|_0^2 = 4\lambda$$

Centroid $(x, y) = \left( \frac{M_y}{\text{mass}}, \frac{M_x}{\text{mass}} \right) = \left( \frac{3}{2}, \frac{6}{5} \right)$

9. Determine the hydrostatic force on the triangle given the density of water $\rho = 1000\text{kg/m}^3$ with a depth $y$ and $g = 9.8\text{m/s}^2$.

See image below for what each term means within the integral

$$F = \int_a^b \rho gd(y) \, dA = \rho g \int_a^b d(y) w(y) \, dy$$

$$F = 9810 \int_{10}^{15} y(y - 10) \, dy$$

$$F = 1635000 \text{ N}$$
10. Consider the curve $y = 5 \ln(x)$ between the points $(1,0)$ and $(e,5)$.

A. SET UP, BUT DO NOT EVALUATE, a $dx$-integral which represents the arc length of the curve.

B. SET UP, BUT DO NOT EVALUATE, a $dy$-integral which represents the arc length of the curve.

C. SET UP, BUT DO NOT EVALUATE, a definite integral which represents the surface area of the surface obtained by rotating the curve around the line $y = 10$.

(a) $ds = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} dx$, $y' = \frac{5}{x}$, $ds = \int_1^e \sqrt{1 + \left( \frac{5}{x} \right)^2}$

(b) $y = 5 \ln(x)$, $e^y = e^{\ln(x)}$, $x = e^y$, $x' = \frac{e^y}{y}$, $ds = \int_0^5 \sqrt{1 + \left( \frac{e^y}{y} \right)^2} dy$

(c) $SA = \int 2\pi y ds$, $ds = \sqrt{1 + \left( \frac{e^y}{5} \right)^2} dy$, $SA = 2\pi \int_0^5 (10 - y) \sqrt{1 + \left( \frac{e^y}{5} \right)^2} dy$
11. The profile \( y = \sqrt{4-x^2} \) on the interval \( x \in [-1, 1] \) is revolved around the \( x \)-axis. Find the surface area of this surface.

Use the following formula to find the surface area of an arc rotated about the \( x \)-axis:

\[
S = \int 2\pi y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx
\]

where \( ds = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \)

We choose to integrate with respect to \( x \) since we are given that interval.

\[
S = \int_{-1}^{1} 2\pi y \sqrt{1 + \left( \frac{-x}{\sqrt{4-x^2}} \right)^2} \, dx
\]

We can simplify the square root term from the \( ds \).

\[
\sqrt{1 + \left( \frac{-x}{\sqrt{4-x^2}} \right)^2} = \sqrt{1 + \frac{x^2}{4-x^2}} = \sqrt{\frac{4-x^2 + x^2}{4-x^2}} = \sqrt{\frac{4}{4-x^2}} = \frac{2}{\sqrt{4-x^2}}
\]

\[
S = \int_{-1}^{1} 2\pi y \frac{2}{\sqrt{4-x^2}} \, dx
\]

We can then write \( y \) in terms of \( x \) and simplify.

\[
S = \int_{-1}^{1} 2\pi \sqrt{4-x^2} \frac{2}{\sqrt{4-x^2}} \, dx
\]

\[
S = \int_{-1}^{1} 4\pi \, dx = 8\pi
\]
12. Compute the arc length of the function $y = 1 + 2x^{\frac{3}{2}}$ between $x = 0$ and $x = 1$

(a) $\frac{14}{9}$
(b) $\frac{10}{9}$
(c) $\frac{2}{9}\sqrt{10}$
(d) $\frac{2}{27}(10\sqrt{10} + 1)$
(e) None of the above

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$y = 1 + 2x^{\frac{3}{2}}, \frac{dy}{dx} = 3x^{\frac{1}{2}}$$

$$S = \int_{0}^{1} \sqrt{1 + (3x^{\frac{1}{2}})^2}dx$$

$$S = \int_{0}^{1} \sqrt{1 + 9x}dx$$

$u = 1 + 9x, du = 9dx, dx = \frac{1}{9}du$

The u-bounds are: $u = 1 + 9(1) = 10, u = 1 + 9(0) = 1$

$$S = \sqrt{u}\frac{1}{9}du$$

$$S = \frac{1}{9}\left(\frac{2}{3}u^{\frac{3}{2}}\right)|_{1}^{10}$$

$$\left(\frac{2}{27}\right)(10^{\frac{3}{2}} - 1^{\frac{3}{2}})$$

$$S = \left(\frac{2}{27}\right)(10\sqrt{10} - 1)$$