The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Session 2:

Can’t make it to a session? Here’s our schedule by course:

https://care.engineering.illinois.edu/tutoring-resources/tutoring-schedule-by-course/

Solutions will be available on our website after the last review session that we host, as well as posted in the zoom chat 30 minutes prior to the end of the session

Step-by-step login for exam review session:

1. Log into Queue @ Illinois
2. Click “New Question”
3. Add your NetID and Name
4. Press “Add to Queue”
5. Join the zoom link in the staff message

Please do not log into the zoom call without adding yourself to the queue

Good luck with your exam!
1. Evaluate the following functions with \( \lim_{(x,y) \to (0,0)} \):

\[
f(x, y) = \frac{3xy - x^2y}{x^2 + y^2 + xy}
\]

Use polar coordinates:

\[
\lim_{(x,y) \to (0,0)} f(x, y) = \lim_{r \to 0} f(r, \theta)
\]

\[
= \lim_{r \to 0} \frac{3r^2 \sin \theta \cos \theta - r^3 \cos^2 \theta \sin \theta}{r^2(\cos^2 \theta + \sin^2 \theta + \cos \theta \sin \theta)}
\]

\[
= \lim_{r \to 0} \frac{3 \sin \theta \cos \theta - r \cos^2 \theta \sin \theta}{(1 + \cos \theta \sin \theta)}
\]

Let \( r \to 0 \) (eliminating the \( \cos(\theta)^2 \sin(\theta) \) term). We end up with

\[
\frac{3 \sin \theta \cos \theta}{1 + \cos \theta \sin \theta}
\]

Plug in \( \theta = 0 \) and \( \theta = \frac{\pi}{4} \)

\[
\begin{align*}
\theta &= 0 \\
0 &= 0
\end{align*}
\]

\[
\begin{align*}
\theta &= \frac{\pi}{4} \\
\frac{3}{2} &= 1
\end{align*}
\]

The limit has different values for different \( \theta \), thus the limit **DOES NOT EXIST**

\[
f(x, y) = \frac{y \sin(x) + y^2 e^x}{y}
\]

Divide out a \( y \), and the let \( (x, y) \to (0,0) \)

\[
\lim_{(x,y) \to (0,0)} \sin(x) + ye^x \to 0
\]

\[
f(x, y) = \frac{(x^2 + y^2)^5}{x^{10} + y^4}
\]

Take the path \( x = 0 \) and you end up with \( f(0, y) = y^6 \), which has a limit of 0

Take the path \( y = 0 \) and you end up with \( f(x, 0) = \frac{x^{10}}{x^m} \) which has a limit of 1

Thus, the limit **DOES NOT EXIST**
2. Consider the region $R$:

(a) Suppose there exists a transformation $S: \mathbb{R}^2 \to \mathbb{R}^2$ from $S$ to $R$. Find $T(u, v)$

It’s convenient to make the substitution $u = xy$ since two functions in $R$ have an $xy$ term.

$$u = xy \in [1, 2]$$

$$v = y \in [1, 2]$$

To solve for $x$, divide by $y$ and then substitute in $v$ for $y$.

$$(x, y) = T(u, v) = \left( \frac{u}{v}, v \right)$$
(b) Use the answer from (a) to evaluate $\int \int_R x^2 dA$

Substitute $\frac{u}{v}$ for $x$ and calculate the Jacobian for this transformation.

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$

Now calculate the integral

$$\int_1^2 \int_1^2 \left(\frac{u}{v}\right)^2 \frac{1}{v} dudv = \frac{7}{8}$$

3. Find the minimum and maximum values of $f(x, y, z) = 3x^2 + 8y^2 + z^3 - 3z$ defined on the domain $x^2 + 4y^2 + 3z \leq 12$ and $z \geq 0$

(a) The domain is (select all that apply)

I) open
II) closed
III) bounded
IV) unbounded

The region includes all of its boundary points, therefore it’s closed and bounded.

(b) Where are the critical points inside the domain? Evaluate the function value on these points.

The critical points inside the domain are where $\nabla f = 0$.

$$\nabla f = \langle 6x, 16y, 3z - 3 \rangle = \langle 0, 0, 0 \rangle$$

Which gives us

$$x = 0, \ y = 0, \ z = 1$$

Critical Points: $f(0, 0, 1) = -2$
(c) What is the minimum and maximum on \( x^2 + 4y^2 + 3z = 12 \)?

Since we’re dealing with the boundary now, we must use Lagrange multipliers. Start by defining \( g(x, y, z) \) as the boundary of the curve with \( g(x, y, z) = 12 \).

\[
g(x, y, z) = x^2 + 4y^2 + 3z
\]

\[
\nabla f = \lambda \nabla g \rightarrow \langle 6x, 16y, 3z - 3 \rangle = \langle 2\lambda x, 8\lambda y, 3\lambda \rangle
\]

Move \( \nabla f \) and \( \lambda \nabla g \) to the same side to solve for each variable.

\[
6x - 2\lambda x = 0 \quad 16y - 8\lambda y = 0 \quad 3z - 3 - 3\lambda = 0
\]

\[
\lambda = 3 \text{ or } x = 0 \quad \lambda = 2 \text{ or } y = 0 \quad \lambda = z - 1
\]

\[
g(x, y, z) = x^2 + 4y^2 + 3z = 12
\]

Solving these four equations for four unknowns, we have four solutions:

\[
\lambda = 3, \ x = \pm 2, \ y = 0, \ z = 0 \quad \lambda = 2, \ x = 0, \ y = \pm \sqrt{3}, \ z = 3
\]

\[
\begin{array}{c}
f(0, 0, 4) = 52 \\
f(0, \pm \frac{\sqrt{3}}{2}, 3) = 24
\end{array}
\]

(d) What is the minimum and maximum on \( z = 0 \)?

We are using a new boundary now, so we have to define a new \( g(x, y, z) \).

\[
g(x, y, z) = z
\]

\[
\nabla f = \lambda \nabla g \rightarrow \langle 6x, 16y, 3z - 3 \rangle = \langle 0, 0, \lambda \rangle
\]

\[
g(x, y, z) = 0
\]

There is one solution \( x = 0, \ y = 0, \ z = 0, \ \lambda = -3 \)

\[
f(0, 0) = 0
\]

(e) What is the global minimum and maximum of the whole domain?

\[
\begin{array}{c}
f(0, 0, 1) = -2 \\
f(0, 0, 4) = 52
\end{array}
\]
4. Let $S$ be a combination of these three surfaces:

\[ S_1 : x^2 + y^2 = 9, \quad 0 < z < 10, \text{ oriented away from } z\text{-axis} \]

\[ S_2 : x^2 + y^2 \leq 9, \quad z = 0, \text{ oriented along negative } z \]

\[ S_3 : \frac{x^2}{9} + \frac{y^2}{9} - (z - 10)^2 \quad z > 10, \text{ oriented along positive } z \]

Let $\vec{F} = \langle x, 1, y \rangle$. Calculate the flux of the vector field through $S$, \( \int_C \vec{F} \cdot \vec{n} \, dS \)

Convert to volume with variables and boundaries:

\[ x = u \cos v \quad y = u \sin v \quad z = z \]

The surfaces become:

\[ u^2 = 9 \quad 0 < z < 10 \]
\[ u^2 \leq 9 \quad z = 0 \]
\[ \frac{u^2}{9} = (z - 10)^2 \]
\[ \frac{u}{3} = (z - 10) \]

Using Divergence Theorem:

\[ \int_0^{2\pi} \int_0^3 \int_0^{\frac{\pi}{2} + 10} \nabla \cdot \vec{F} \, u \, dz \, du \, dv = \int_0^{2\pi} \int_0^3 (10 - \frac{u}{3}) \, u \, du \, dv \]

\[ 84\pi \]
5. For an outward facing surface $S$ of the paraboloid $x^2 - y + 2z^2 = 0$ in the region $y \leq 2$, find $\iint_S \nabla \times \vec{F} \cdot d\vec{S}$ with:

$$\vec{F} = \langle z, x, 0 \rangle$$

Use Stokes Theorem. The boundary with proper orientation is the ellipse $C$ given by:

$$x = -\sqrt{2} \cos u$$
$$y = 2$$
$$z = \sin u$$

Solving for $\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle z, x, 0 \rangle \cdot \langle -\sqrt{2} \cos u, 2 \sin u \rangle \sqrt{2}\pi$
6. Consider the region bounded by a sphere $x^2 + y^2 + z^2 = 1$ and a cone $z = \sqrt{3x^2 + 3y^2}$ above the $xy$ plane. Find the volume of the region.

Spherical coordinates is most useful here since the figure resembles part of a sphere. Remember that the Jacobian for spherical coordinates is $\rho^2 \sin(\phi)$.

The bounds on $\theta$ and $\rho$ are easy to find: $0 \leq \theta \leq 2\pi$ and $0 \leq \rho \leq 1$. The bounds for $\phi$ require converting the functions into spherical functions and solving for $\phi$.

For the sphere, this is $\rho = 1$

For the cone, this is $\rho^2 \cos^2(\phi) = 3\rho^2 \sin^2(\phi) \cos^2(\theta) + 3\rho^2 \sin^2(\phi) \sin^2(\theta)$

All the $\rho^2$ terms cancel, and $\cos^2(\theta) + \sin^2(\theta)$ reduces to 1, leaving us with $\cos^2(\phi) = 3 \sin^2(\phi)$

Then solving for $\phi$, we get $\phi = \frac{\pi}{6}$

Thus the bounds are $0 \leq \phi \leq \frac{\pi}{6}$

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^1 \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta$$

$$\frac{1}{3}(1)^3 \left( - \cos \frac{\pi}{6} + \cos 0 \right)(2\pi)$$

$$\frac{1}{3}(2 - \sqrt{3})\pi$$