The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Nov 2, 6-8pm: Danny, Pieter, Matthew
Session 2: Nov 3, 7-9pm: David, Nidhi, Sean

Can’t make it to a session? Here’s our schedule by course:
https://care.engineering.illinois.edu/tutoring-resources/tutoring-schedule-by-course/

Solutions will be available on our website after the last review session that we host, as well as posted in the zoom chat 30 minutes prior to the end of the session.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois
2. Click “New Question”
3. Add your NetID and Name
4. Press “Add to Queue”
5. Join the zoom link in the staff message

Please do not log into the zoom call without adding yourself to the queue

Good luck with your exam!
1. The vector field $\vec{F} = \langle 2xy + 2x + y^2, 2xy + 2y + x^2 \rangle$ is conservative. Find a potential function $f$ for $\vec{F}$ (a function with $\nabla f = \vec{F}$)

$$\vec{F} = \langle f_x, f_y \rangle = \langle 2xy + 2x + y^2, 2xy + 2y + x^2 \rangle$$

$$\int f_x = yx^2 + x^2 + xy^2 + C(y)$$

$$\int f_y = xy^2 + y^2 + yx^2 + C(x)$$

Looking at all the terms and comparing with $\vec{F}$, we know that $C(y) = y^2$ and $C(x) = x^2$, therefore the potential function is:

$$f(x, y, z) = yx^2 + x^2 + xy^2 + y^2$$
2. A particle moves along the upper part of an ellipse in the $xy$-plane that has its center at the origin with semi-major and semi-minor axes $a = 4$ and $b = 3$, respectively. Starting at $(a, 0)$ and ending at $(-a, 0)$ and subject to the following force field, what is the total work done?

$$\vec{F} = (3x - 4y + 2z)\hat{i} + (4x + 2y - 3z^2)\hat{j} + (2xz - 4y^2 + z^3)\hat{k}$$

(1) Find the parameterization of the ellipse

$$x = 4 \cos t, \ y = 3 \sin t, \ z = 0$$

$$dx = -4 \sin t \, dt, \ dy = 3 \cos t \, dt, \ dz = 0$$

Recall that $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

(2) Take the dot product in the line integral for work. (The $z$ component is zero so it can be ignored here).

$$\oint \vec{F} \cdot d\vec{r} = \int [(3x - 4y)\hat{i} + (4x + 2y)\hat{j}] \cdot (dx\hat{i} + dy\hat{j})$$

$$= \int (3x - 4y)dx + (4x + 2y)dy$$

(3) Substitute in the parameterization found in (1)

$$\int (12 \cos t - 12 \sin t)(-4 \sin t)dt + (16 \cos t + 6 \sin t)(3 \cos t)dt$$

(4) Determine the times that the particle is at its starting and ending position, which in this case is $0 < t < \pi$. And solve the integral

$$\int_{t=0}^{t=\pi} \left[(12 \cos t - 12 \sin t)(-4 \sin t) + (16 \cos t + 6 \sin t)(3 \cos t)\right] dt$$

$$= \frac{48\pi}{3}$$
3. Compute the work done by the force field \( \vec{F} = x \hat{i} + 3xy \hat{j} - (x + z) \hat{k} \) as a particle moves in a line segment from \((1, 4, 2)\) to \((0, 5, 1)\).

\[
W = \int_C \mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r}
\]

First, parameterize the line as \( \mathbf{r}(t) = (1, 4, 2) + (-1, 1, -1)t \) with \( 0 \leq t \leq 1 \) and plug it into the vector field to get

\[
\mathbf{F} = (1-t)\hat{i} + 3(1-t)(4+t)\hat{j} - (1-t+2-t)\hat{k}
\]

Now find \( d\mathbf{r} = (-1, 1, -1) \) and take the dot product with the vector field

\[
\mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r} = 14 - 10t - 3t^2 \, dt
\]

And compute the integral

\[
W = \int_C \mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r}
\]

\[
\int_0^1 14 - 10t - 3t^2 \, dt = 8
\]

4. Let \( r(t) = (\sin(t), \cos^2(t)), 0 \leq t \leq 2\pi \). Which graph below represents this curve?

Converting this into a Cartesian equation gives

\[
y + x^2 = 1 \implies y = 1 - x^2
\]

Which is a concave down parabola with its vertex at \((0, 1)\)
5. Consider the following vector fields $\vec{F}(x, y, z)$. Are they conservative? If so, find a function $f(x, y, z)$ so that $\nabla f = \vec{F}$. If not, justify your response.

**Conservative vector field test:** A vector field $\vec{F}$ is conservative if the curl is the zero vector.

$$\vec{\nabla} \times \vec{F} = \left| \begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{array} \right| = \langle \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial z}, \frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} \rangle = \vec{0}$$

a) $\vec{F}(x, y, z) = \langle yz, xz, xy + 2z \rangle$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy + 2z \end{vmatrix} = \langle x - x, y - y, z - z \rangle = \vec{0}$$

The vector field is conservative, therefore, a potential function exists. To find it, we must find the necessary terms from each component (We neglect the constant for now, we’ll add it back later).

$$\int F_x \, dx = \int yz \, dx = xyz$$
$$\int F_y \, dy = \int xz \, dy = xyz$$
$$\int F_z \, dz = \int xy + 2z \, dz = xyz + z^2$$

We see that the necessary terms are $xyz$ and $z^2$, therefore

The field is conservative and has potential function $f(x, y, z) = xyz + z^2 + C$
b) \( \vec{F}(x, y, z) = (y + e^x, x - \cos y, 4 + z) \)

\[
\begin{vmatrix}
\frac{\hat{x}}{\partial x} & \frac{\hat{y}}{\partial y} & \frac{\hat{z}}{\partial z} \\
y + e^x & x - \cos y & 4 + z
\end{vmatrix}
= (0 - 0 - 0, 1 - 1) = 0
\]

The vector field is conservative, therefore, a potential function exists. To find it, we must integrate each component (We neglect the constant for now, we’ll add it back later).

\[
\begin{align*}
\int F_x \, dx &= \int (y + e^x) \, dx = xy + e^x \\
\int F_y \, dy &= \int (x - \cos y) \, dy = xy - \sin y \\
\int F_z \, dz &= \int (4 + z) \, dz = 4z + \frac{1}{2}z^2
\end{align*}
\]

We see that the necessary terms are \(xy, e^x, -\sin y, 4z,\) and \(\frac{1}{2}z^2\), therefore:

The field is conservative and has potential function \( f(x, y, z) = xy + e^x - \sin y + 4z + \frac{1}{2}z^2 \).

c) \( \vec{F}(x, y, z) = (y, z^2, x) \)

\[
\begin{vmatrix}
\frac{\hat{x}}{\partial x} & \frac{\hat{y}}{\partial y} & \frac{\hat{z}}{\partial z} \\
y & z^2 & x
\end{vmatrix}
= (-2z, 1, -1)
\]

This vector field is not conservative. Therefore, a potential function does not exist.
6. Find the work done by the force field below in moving an object from (1,1) to (2,4) (HINT: Check if the vector field is conservative).

\[ \vec{F}(x, y) = \langle 6y^{\frac{3}{2}}, 9x\sqrt{y} \rangle \]

First we need to check if this vector field is conservative

\[ \frac{\partial F_x}{\partial y} = 9\sqrt{y} \quad \text{and} \quad \frac{\partial F_y}{\partial x} = 9\sqrt{y} \]

Since \( \frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} \) we can say the vector field is conservative.

Now we can find a function \( f(x, y) \) such that \( \nabla f = \vec{F} \)

\[
\begin{align*}
\int F_x \, dx &= \int 6y^{\frac{3}{2}} \, dx = 6xy^{\frac{3}{2}} \\
\int F_y \, dy &= \int 9x\sqrt{y} \, dy = 6xy^{\frac{5}{2}}
\end{align*}
\]

Thus our potential function is

\[ f(x, y) = 6xy^{\frac{3}{2}} + C \]

Now that we have a potential function we can use the Fundamental Theorem of Line Integrals to compute the work done in moving from (1,1) to (2,4)

\[
W = \int_C \vec{F} \cdot dr = f(x_2, y_2) - f(x_1, y_1) = \\
f(2, 4) - f(1, 1) = (96 + C) - (6 + C)
\]

\[ W = 90 \text{ (units)} \]
7. Compute the double integral over the indicated rectangle. Confirm your answer by switching the order of integration and recomputing.

\[ \int \int_{R} 2x - 4y^3 \, dA \quad R = [-5, 4] \times [0, 3] \]

\[ \int_{0}^{3} \int_{-5}^{4} 2x - 4y^3 \, dx \, dy \]
\[ \int_{0}^{3} -9 - 36y^2 \, dy \]
\[ -9y - 9y^4 \Big|_{0}^{3} = -756 \]

\[ \int_{-5}^{4} \int_{0}^{3} 2x - 4y^3 \, dy \, dx \]
\[ \int_{-5}^{4} 6x - 81 \, dx \]
\[ 3x^2 - 81x \Big|_{-5}^{4} = -756 \]

8. Making an appropriate change of variables, compute the following double integral over the region bound by a circle of radius 2 and a circle of radius 5.

\[ \int \int_{D} e^{x^2+y^2} \, dA \]

Using polar coordinates gives the following integral (don’t forget the Jacobian)

\[ \int \int_{D} e^{x^2+y^2} \, dA = \int_{0}^{2\pi} \int_{2}^{5} r e^{r^2} \, dr \, d\theta \]

Then using a \( u \)-substitution \( u = r^2 \) for the \( r \) dependence

\[ \frac{1}{2} \int_{0}^{2\pi} \int_{4}^{25} e^u \, dud\theta = \pi (e^{25} - e^4) \approx 2.26 \times 10^{11} \]