Assortment Optimization under Markov Chain Choice Model for Multi-Category Products

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Abstract

In this research, we explored the assortment optimization for multi-category products. We built a Markov chain choice model to model customers' choices in a two-category retail systems and generalize the correlation within and between categories as state transition. Based on the asymmetric cross-selling effect, we classified the two categories by the primary category and secondary category and assumed that both the initial interests and transition of interests in the the secondary category products depend on the purchase decision in the primary category, but the reversed case does not hold. Under these assumptions, we adapted the expectation-maximization algorithm to estimate the parameters for the proposed Markov chain choice model and formulated a linear program to solve for an optimal assortment. The numerical experiments demonstrate that there is about 1% to 5% improvement in revenue by the proposed model compared with the benchmark, the Markov chain choice model for independent choice across different categories.

1 Introduction

Revenue management is the subject for commercials how to make better decision to improve their revenue. Assortment, the collection of products offered to customers, is one of the decisions a commercial need to take into consideration. To find an assortment, customers' choices play an important role since these choices determine the demand of each product. Given different collections of offered products, customers make different purchase decisions so the total revenue will be very different under different assortments. The problem of finding an optimal assortment attracts more and more researchers and many models are developed. We care about models for customer choice behavior, how to fit the model from observed sales records, and the assortment optimization formulation under the proposed model. There are already plentiful constructive results and progresses in assortment optimization, which will be discussed in the later section.

In this research, we focus on the multi-category products and assume that there is choice correlation not only within each category, but also between multiple categories. For the simplest two-category case, we study a cross-selling effect between these two categories called asymmetric effect, where these two categories are characterized by a primary category and a secondary category. This asymmetric cross-selling effect is that when a customer makes a decision purchasing or not among the primary category products, he or she may consider a product of the secondary category, but this process cannot happen reversely. It is unlikely that the secondary category products have significant effect on the primary category products such that a customer move the willingness to purchase from the secondary category to the primary category. One example is the spaghetti and spaghetti sauce mentioned above, and another example is the phone and phone case. It is reasonable that one arrives for a new phone and then have interest in a new phone case, but the reverse case makes no sense.

We build models to study how customers select their preferred products and make purchase decisions on products of two different categories under this asymmetric cross-selling setting. Since we are considering the path of a customers decision-making, we can treat it as a Markov chain with the assumption of the Markov property in transition of interests. Therefore, the choice model is based on the Markov chain choice model.

1.1 Literature Review

Chong et al. (2001) suggests that customers will form a order of products based on various features. When the highest priority product is not offered, the second priority product will play a role of substitution. The Markov chain choice model for one category products is proposed by Blanchet et al. (2016). They show that many popular customer choice models can be exactly expressed by the Markov chain choice model so that Markov chain choice model can be generalized in many business settings. Also, they show that the assortment optimization under Markov chain choice model can be solved efficiently by providing a polynomial-time algorithm to find the optimal assortment exactly. Desir et al. (2015) study constrained assortment optimization problems subject to capacity constraints under the Markov chain choice model. Markov chain choice model is a flexible model and widely used in assortment optimization.

The expectation-maximization algorithm is first proposed by Dempster et al. (1977), which is designed to deal with the incomplete log-likelihood due to missing data and estimate the parameters by maximizing a carefully constructed complete log-likelihood. The estimated parameters from the complete log-likelihood are guaranteed to converge to a local maximum of the incomplete log-likelihood function when some regularity conditions are satisfied, which is proved by Wu (1983) and Nettleton (1999). Simsek and Topaloglu (2018) use the expectation-maximization algorithm to estimate the parameters in this Markov chain choice model.

On the other hand, when it comes to the multi-category case, things become complicated. The asymmetric cross-selling effect was discussed by Walters (1991) with the example of spaghetti and spaghetti sauce. However, the choice models under this asymmetric cross-selling effect and corresponding optimization formulations are not well-studied. Ghoniem et al. (2016) studied the assortment and price optimization under this asymmetric cross-selling effects a maximum-surplus choice model. However, this model is based on strong assumption on the price of products, which cannot solve for general multi-category assortment optimization problems. Therefore, it is a natural idea to use the Markov chain choice model for the customers' choices and assortment optimization.

1.2 Main Contributions

We extend the Markov chain choice model for the multi-category assortment optimization problems. We adapt the expectation-maximization algorithm to estimate the parameters. We also formulate a linear program to solve for the optimal assortment. Furthermore, after we figure out the Markov chain choice model for a two-category system with this asymmetric cross-selling effect, it is reasonable that we can use this asymmetric cross-selling effect to construct a hierarchical structure of products and formulate a larger Markov chain choice model for it. For example, in an electronic store, we can find plentiful such effects, including monitors and Nintendo Switch consoles, consoles and games, and consoles and corresponding accessories. Then we have a hierarchical structure.

1.3 Outline

In section 2, we describe how the Markov chain choice model is built for the multi-category products setting. Also, the common notations in this article will be defined in this section. In section 3, we derive the expectation-maximization algorithm to estimate the parameters in this multi-category Markov chain choice model and provide the proof for the convergence of the proposed expectation-maximization algorithm. In section 4, we formulate a linear program for assortment optimization under the proposed multi-category Markov chain choice model and prove the optimality of the solution. In section 5, we present the numerical experiments to show the performance of our model. In section 6, we discuss our ongoing work, a further assortment optimization formulation by robust optimization.

2 Model Formulation

Denote two categories M and N, where category M is the primary and purchase interest in category M can spark purchase interest in the secondary category N. There are $|M|$ products in category M, denoted by product $m_1, m_2, \cdots, m_{|M|}$. Similarly, there are $|N|$ products in category N, denoted by product $n_1, n_2, \cdots, n_{|N|}$. The no purchase options in the two categories are denoted by m_0 and n_0 and the categories with no purchase option are denoted by M_+ and N_+. From these two categories, collections S_M and S_N are offered. Then not offered sets are denoted by $\bar{S}_M$ and $\bar{S}_N$. When the full set of category M is offered, the vector of probabilities that a customer arrives with purchase willingness on each product in category M is denoted by $\mathbf{\Lambda}_M$, and each probability of interest in product m_j is denoted by λ_{m_j} $\forall m_j \in M_+$. We have the similar notation $\mathbf{\Lambda}_N$ and λ_{n_j} $\forall n_j \in N_+$ for category N. Then we denote the transition probability by a transition matrices $\mathbf{P}_M$ and $\mathbf{P}_N$ by

$$\mathbf{P}_M = \begin{bmatrix} \rho_{m_0 m_0} & \rho_{m_0 m_1} & \cdots & \rho_{m_0 m_{|M|}} \\ \rho_{m_1 m_0} & \rho_{m_1 m_1} & \cdots & \rho_{m_1 m_{|M|}} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{m_{|M|} m_0} & \rho_{m_{|M|} m_1} & \cdots & \rho_{m_{|M|} m_{|M|}} \end{bmatrix} \text{ and } \mathbf{P}_N = \begin{bmatrix} \rho_{n_0 n_0} & \rho_{n_0 n_1} & \cdots & \rho_{n_0 n_{|N|}} \\ \rho_{n_1 n_0} & \rho_{n_1 n_1} & \cdots & \rho_{n_1 n_{|N|}} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n_{|N|} n_0} & \rho_{n_{|N|} n_1} & \cdots & \rho_{n_{|N|} n_{|N|}} \end{bmatrix}.$$

In this matrix, the probability of interest transition from product m_i to product m_j is given by $\rho_{m_i m_j}$ in the primary category and that from product n_i to product n_j is given by $\rho_{n_i n_j}$ in the secondary category. Also, we denote the "spark" possibility of product m_j in the secondary category by product n_j in the primary category by $\sigma_{m_j n_j}$. Therefore, we can also construct a spark matrix $\mathbf{S}$ by

$$\mathbf{S} = \begin{bmatrix} \sigma_{m_0 n_0} & \sigma_{m_0 n_1} & \cdots & \sigma_{m_0 n_{|N|}} \\ \sigma_{m_1 n_0} & \sigma_{m_1 n_1} & \cdots & \sigma_{m_1 n_{|N|}} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{m_{|M|} n_0} & \sigma_{m_{|M|} n_1} & \cdots & \sigma_{m_{|M|} n_{|N|}} \end{bmatrix}$$

According to our definition of primary category and secondary category, we do not allow purchase interest in product $m_j \in M_+$ sparked by product $n_j \in N_+$. Therefore, the purchase interest in category M products can only be transferred within category M and cannot be sparked by purchase decision in category N. We can keep using the Markov chain choice model for one category proposed by Blanchet et al. (2016). We denote the probability that a customer in this two-category system with S_M offered considers product m_j by

$$\phi_{m_j}(S_M) = \lambda_{m_j} + \sum_{m_i \in \bar{S}_M} \rho_{m_i m_j} \phi_{m_i}(S_M), \ \forall m_j \in M_+.$$

The probability of considering each product in category M can be obtained by solving a system of linear equation. Define $\mathbf{\Phi}_{\bar{S}_M}(S_M) = \left\{\phi_{m_j}(S_M) \mid m_j \in \bar{S}_M\right\}$, $\mathbf{\Lambda}_{\bar{S}_M} = \left\{\lambda_{m_j} \mid m_j \in \bar{S}_M\right\}$, and $\mathbf{P}_{\bar{S}_M} = \left\{\rho_{m_i m_j} \mid m_i, m_j \in \bar{S}_M\right\}$. Then we have a system of linear equations $\left(\mathbf{I} - \mathbf{P}_{\bar{S}_M}\right)^{\mathrm{T}} \mathbf{\Phi}_{\bar{S}_M}(S_M) = \mathbf{\Lambda}_{\bar{S}_M}$. According to Puterman (1994, corollary C.4), if the probability of interest transition $\rho_{m_i m_j} > 0 \ \forall m_i, m_j \in M_+$ is satisfied, then $\left(\mathbf{I} - \mathbf{P}_{\bar{S}_M}\right)^{-1}$ exists for any assortment S_M. Therefore, we always have the solution for the system of linear equations by

$$\mathbf{\Phi}_{\bar{S}_M}(S_M) = \left(\mathbf{I} - \mathbf{P}_{\bar{S}_M}\right)^{-\mathrm{T}} \mathbf{\Lambda}_{\bar{S}_M}.$$

With this solution, we can easily calculate the probability of visiting each product in category M. Then we want to determine the probability of visiting product n_j, which needs to be discussed in two cases. When a customer only wants to purchase a product from category N products, we just treat it as the single-category case. When a customer is initially interested in category M products and then interested in category N products, this sparked interest depends on the previous purchase decision on category M products, so we need to consider a conditional probability $\phi_{n_j|m_k}(S_M, S_N)$ that a customer in this two-category system with S_M and S_N offered considers product n_j given a purchase decision on product m_j. For secondary category N, the initial purchase interest in product n_j is sparked by a purchase decision of product m_j in category M with probability $\lambda_{n_j|m_j} = \sigma_{m_j n_j}$. We consider the purchase interests transferred within category N with the assumption that the transition of interests among the second category products is correlated to the purchase decision in the primary category. This assumption makes sense in many business settings. For example, when a customer bought a set of suit and considered to buy a pair of leather shoes then, he might be more likely to buy a pair of shoes matching his suit. Therefore, he might narrow down the choice of colors and styles. To address with this correlation, we denote the purchase interests transferred from product n_i to product n_j in category N conditioned on the purchase decision on product m_j in category M with probability $\rho_{n_i n_j|m_j} \phi_{n_i|m_j}(S_M, S_N)$ and the transition matrix in the category N by $\mathbf{P}_{N|m_j}$ for each purchase decision on product m_j in category M. Therefore, we consider all the probability that a customer in this two-category system with S_M and S_N offered considers product n_j given a purchase decision on product m_j by

$$\phi_{n_j|m_j}(S_M,S_N)=\sigma_{m_jn_j}+\sum_{n_i\in\bar{S}_N}\rho_{n_in_j|m_j}\phi_{n_i|m_j}(S_M,S_N),\ \forall n_j\in N_+.$$

Then each conditional probability above can be calculated by solving a system of linear equations. We define $\mathbf{\Phi}_{\bar{S}_N|m_j}(S_M,S_N)=\left\{\phi_{n_j|m_j}(S_M,S_N)\mid n_j\in\bar{S}_N\right\}$ and $\mathbf{S}_{m_j\bar{S}_N}=\left\{\sigma_{m_jn_j}\mid n_j\in\bar{S}_N\right\}$. If the probability of interest transition $\rho_{n_in_j|m_j}>0\ \forall n_i,n_j\in N_+$ is satisfied, then $\left(\mathbf{I}-\mathbf{P}_{\bar{S}_N|m_j}\right)^{-1}$ exists for any assortment S_M and S_N. So we have the solution by

$$\mathbf{\Phi}_{\bar{S}_N|m_j}(S_M,S_N)=\left(\mathbf{I}-\mathbf{P}_{\bar{S}_N|m_j}\right)^{-\mathrm{T}}\mathbf{S}^{\mathrm{T}}_{m_j\bar{S}_N}.$$

When we have each conditional probability, we can easily calculate the joint probability $\phi_{m_jn_j}(S_M,S_N)$ that a customer considers any product combination of m_j and n_j when S_M and S_N are offered. Define a collection of conditional transition matrices within category N that $\mathbf{P}_{N|M}=\left\{\mathbf{P}_{N|m_j}\mid m_j\in M_+\right\}$. Since we easily see that we can obtain all the information in this model after we know the parameters $\mathbf{\Lambda}_M$, $\mathbf{P}_M$, $\mathbf{P}_{N|M}$ and $\mathbf{S}$, the next question is how to estimate these parameters. This will be used to calculate the likelihood of parameters and all the detailed procedures will be explained in detail in the next section.

3 Parameter Estimation

To further optimize the assortment selection, we need to estimate the parameters in the proposed Markov chain choice model. We will utilize the expectation-maximization based estimation method proposed by Simsek and Topaloglu (2018) and extend the model to the multi-category case. Denote customer t's purchase decision on category M products under assortment S_M by random variable $\mathbf{Z}_M\left(S_M^{(t)}\right)$ that

$$\mathbf{Z}_M\left(S_M^{(t)}\right)=\left(Z_{m_0}\left(S_M^{(t)}\right),Z_{m_1}\left(S_M^{(t)}\right),\ldots,Z_{m_{|M|}}\left(S_M^{(t)}\right)\right)\in\left\{\mathbf{e}_{m_0},\mathbf{e}_{m_1},\ldots,\mathbf{e}_{m_{|M|}}\right\},$$

where $\mathbf{e}_{m_j}\in\{0,1\}^{|M|+1}$ and only the entry corresponding to product m_j is 1. When customer t purchases product m_j under assortment S_M, we have $Z_{m_j}\left(S_M^{(t)}\right)=1$ and $Z_{m_i}\left(S_M^{(t)}\right)=0\ \forall m_i\in M_+\setminus\{m_j\}$. Therefore, we have $\mathbf{Z}_M\left(S_M^{(t)}\right)=\mathbf{e}_{m_j}$. Moreover, we know that $\mathbb{P}\left\{\mathbf{Z}_M\left(S_M^{(t)}\right)=\mathbf{e}_{m_j}\right\}=\phi_{m_j}\left(S_M^{(t)}\right)$ for any offered product $m_j\in S_M$. Similarly, denote customer t's purchase decision on category N products given the purchase decision in category M under assortment S_M and S_N by $\mathbf{Z}_{N|M}\left(S_M^{(t)},S_N^{(t)}\right)$ that

$$\mathbf{Z}_{N|M}\left(S_M^{(t)},S_N^{(t)}\right)=\begin{bmatrix}Z_{n_0|m_0}\left(S_M^{(t)},S_N^{(t)}\right) & Z_{n_1|m_0}\left(S_M^{(t)},S_N^{(t)}\right) & \cdots & Z_{n_{|N|}|m_0}\left(S_M^{(t)},S_N^{(t)}\right)\\ Z_{n_0|m_1}\left(S_M^{(t)},S_N^{(t)}\right) & Z_{n_1|m_1}\left(S_M^{(t)},S_N^{(t)}\right) & \cdots & Z_{n_{|N|}|m_1}\left(S_M^{(t)},S_N^{(t)}\right)\\ \vdots & \vdots & \ddots & \vdots\\ Z_{n_0|m_{|M|}}\left(S_M^{(t)},S_N^{(t)}\right) & Z_{n_1|m_{|M|}}\left(S_M^{(t)},S_N^{(t)}\right) & \cdots & Z_{n_{|N|}|m_{|M|}}\left(S_M^{(t)},S_N^{(t)}\right)\end{bmatrix}.$$

Now we can define the row of $\mathbf{Z}_{N|M}\left(S_M^{(t)},S_N^{(t)}\right)$ corresponding to the given purchase decision on the product m_j by $\mathbf{Z}_{N|m_j}\left(S_M^{(t)},S_N^{(t)}\right)\in\left\{\mathbf{e}_{n_0},\mathbf{e}_{n_1},\ldots,\mathbf{e}_{n_{|N|}}\right\}$, where $\mathbf{e}_{n_j}\in\{0,1\}^{|N|+1}$ and only the entry corresponding to product n_j is 1. When customer t purchases product n_j under assortment S_M and S_N given the previous purchase decision on product m_j, we have $Z_{n_j|m_j}\left(S_M^{(t)},S_N^{(t)}\right)=1$ and $Z_{n_i|m_j}\left(S_M^{(t)},S_N^{(t)}\right)=0\ \forall n_i\in N_+\setminus\{n_j\}$. Therefore, we have $\mathbf{Z}_{N|m_j}\left(S_M^{(t)},S_N^{(t)}\right)=\mathbf{e}_{n_j}$. Moreover, we know that $\mathbb{P}\left\{\mathbf{Z}_{N|m_j}\left(S_M^{(t)},S_N^{(t)}\right)=\mathbf{e}_{n_j}\right\}=\phi_{n_j|m_j}\left(S_M^{(t)},S_N^{(t)}\right)$ for any offered product $n_j\in S_N$. From empirical data, we have the assortment $\hat{S}_M^{(t)}$ and $S_N^{(t)}$ for each customer t. Also, for each pair of assortment, we have the customer t's purchase decision $\hat{\mathbf{Z}}_M^{(t)}\sim\mathbf{Z}_M\left(S_M^{(t)}\right)$ and

$\hat{\mathbf{Z}}_{N|m_j}^{(t)} \sim \mathbf{Z}_{N|m_j}\left(S_M^{(t)}, S_N^{(t)}\right)$. Denote the parameters $\left(\mathbf{\Lambda}_M, \mathbf{P}_M, \mathbf{P}_{N|M}, \mathbf{S}\right)$ by $\mathbf{W}$. Therefore, we can write a likelihood function for customer t's purchase decision by

$$L^{(t)}(\mathbf{W}) = \prod_{m_j \in M_+} \left[\phi_{m_j}\left(\hat{S}_M^{(t)} \mid \mathbf{\Lambda}_M, \mathbf{P}_M\right) \prod_{n_j \in N_+} \phi_{n_j|m_j}\left(\hat{S}_M^{(t)}, \hat{S}_N^{(t)} \mid \mathbf{W}\right)^{\hat{Z}_{n_j|m_j}^{(t)}}\right]^{\hat{Z}_{m_j}^{(t)}}.$$

We can consider the likelihood of the purchase decisions all the customers by $L(\mathbf{W}) = \prod_{t \in T} L^{(t)}(\mathbf{W})$. Then we have the log-likelihood for empirical data $\left\{\left(\hat{S}_M^{(t)}, \hat{S}_N^{(t)}, \hat{\mathbf{Z}}_M^{(t)}, \hat{\mathbf{Z}}_{N|m_j}^{(t)}\right) \mid t \in T\right\}$ by $l(\mathbf{W}) = \sum_{t \in T} l^{(t)}(\mathbf{W})$, where

$$l^{(t)}(\mathbf{W}) = \sum_{m_j \in M_+} Z_{m_j}^{(t)} \left[\log \phi_{m_j}\left(\hat{S}_M^{(t)} \mid \mathbf{\Lambda}_M, \mathbf{P}_M\right) + \sum_{n_j \in N_+} Z_{n_j|m_j}^{(t)} \log \phi_{n_j|m_j}\left(\hat{S}_M^{(t)}, \hat{S}_N^{(t)} \mid \mathbf{W}\right)\right].$$

We know that the optimal feasible parameters that fit the given data maximize this log-likelihood function. Therefore, to estimate the parameters, we can formulate and solve an optimization problem that

$$\begin{aligned} \max_{\mathbf{W}} \quad & l(\mathbf{W}) \\ \text{s.t.} \quad & \sum_{m_j \in M_+} \lambda_{m_j} = 1, \\ & \sum_{m_j \in M_+} \rho_{m_i m_j} = 1 \quad \forall m_i \in M_+, \\ & \sum_{n_j \in N_+} \rho_{n_i n_j | m_j} = 1 \quad \forall m_j \in M_+,\ n_i \in N_+, \\ & \sum_{n_j \in N_+} \sigma_{m_j n_j} = 1 \quad \forall m_j \in M_+. \end{aligned} \tag{1}$$

However, according to Simsek and Topaloglu (2018), this optimization problem is hard to solve since there is no closed-form expression for $\phi_{m_j}\left(\hat{S}_M^{(t)} \mid \mathbf{\Lambda}_M, \mathbf{P}_M\right)$ and $\phi_{n_j|m_j}(\mathbf{W})$. This difficulty motivates to use an expectation-maximization algorithm.

3.1 Overview of Expectation-Maximization Algorithm

The general idea of an expectation-maximization algorithm is that we initialize the required parameters and update them iteratively through an expectation step and a maximization step. We need to define some new random variables to construct a new log-likelihood function, estimate the expectation of the new random variables conditioned on previous estimated parameters, and then update the parameters by maximizing the new log-likelihood function. We define the initial interest in category M products to be our first new random variable. We denote this random variable by

$$\mathbf{F}_M = \left(F_{m_0}, F_{m_1}, \ldots, F_{m_{|M|}}\right) \in \left\{\mathbf{e}_{m_0}, \mathbf{e}_{m_1}, \ldots, \mathbf{e}_{m_{|M|}}\right\}.$$

When customer t enters into the system with initial interest in product m_j , we have $F_{m_j} = 1$ and $F_{m_i} = 0\ \forall m_i \in M_+ \backslash \{m_j\}$. Therefore, we have $\mathbf{F}_M = \mathbf{e}_{m_j}$. Moreover, by the definition of initial interest we have that $\mathbb{P}\left\{\mathbf{F}_M = \mathbf{e}_{m_j}\right\} = \lambda_{m_j}$. Then we define the interest transition of customer t within category M products under assortment S_M by

$$\mathbf{G}_M\left(S_M^{(t)}\right) = \begin{bmatrix} G_{m_0 m_0}\left(S_M^{(t)}\right) & G_{m_0 m_1}\left(S_M^{(t)}\right) & \cdots & G_{m_0 m_{|M|}}\left(S_M^{(t)}\right) \\ G_{m_1 m_0}\left(S_M^{(t)}\right) & G_{m_1 m_1}\left(S_M^{(t)}\right) & \cdots & G_{m_1 m_{|M|}}\left(S_M^{(t)}\right) \\ \vdots & \vdots & \ddots & \vdots \\ G_{m_{|M|} m_0}\left(S_M^{(t)}\right) & G_{m_{|M|} m_1}\left(S_M^{(t)}\right) & \cdots & G_{m_{|M|} m_{|M|}}\left(S_M^{(t)}\right) \end{bmatrix}.$$

Since for each product m_i, the purchase interest must transfer and can only transfer to another product m_j, we denote the row of $\mathbf{G}_M\left(S_M^{(t)}\right)$ by $\mathbf{G}_{m_i}\left(S_M^{(t)}\right)$ corresponding to the purchase interest transferred from product m_i. When customer t transfer his purchase interest from product m_i to product m_j with probability $g_{m_im_j}$, we have that $\mathbb{P}\left\{\mathbf{G}_{m_i}\left(S_M^{(t)}\right)=\mathbf{e}_{m_j}\right\}=g_{m_im_j}$. We define a collection of random variables $\left\{\mathbf{G}_{m_i}\left(S_M^{(t)}\right)\mid m_i\in M_+\right\}$. Similarly, we can define the interest transition of customer t within category N products conditioned on the purchase decision on product m_j under assortment S_M and S_N by $\mathbf{G}_{N|m_j}\left(S_M^{(t)},S_N^{(t)}\right)$. Also, the row of $\mathbf{G}_{N|m_j}\left(S_M^{(t)},S_N^{(t)}\right)$ corresponding to the purchase interest transferred from product n_i is denoted by $\mathbf{G}_{n_i|m_j}\left(S_M^{(t)},S_N^{(t)}\right)$. We need to be careful that only when product m_j is offered, we have that $\mathbf{G}_{n_i|m_j}\left(S_M^{(t)},S_N^{(t)}\right)\in\left\{\mathbf{e}_{n_0},\mathbf{e}_{n_1},\ldots,\mathbf{e}_{n_{|N|}}\right\}$. When product m_j is not offered, the transition from product m_j to any product n_j is not a feasible path in this Markov chain choice model. Therefore, we have that $\mathbf{G}_{n_i|m_j}\left(S_M^{(t)},S_N^{(t)}\right)$ does not exist. To eliminate this infeasibility and keep the format, we can set $\mathbb{P}\left\{\mathbf{G}_{n_i|m_j}\left(S_M^{(t)},S_N^{(t)}\right)=\mathbf{0}\right\}=1\ \forall m_j\in\bar{S}_M$. We define a collection of random variables $\left\{\mathbf{G}_{n_i|m_j}\left(S_M^{(t)},S_N^{(t)}\right)\mid m_j\in M_+,\ n_i\in N_+\right\}$. Finally, we need to consider the purchase interest in category N products sparked by category M products. We define customer t's interest in category N products sparked by category M products under assortment S_M by

$$\mathbf{H}_M\left(S_M^{(t)}\right)=\begin{bmatrix} H_{m_0n_0}\left(S_M^{(t)}\right) & H_{m_0n_1}\left(S_M^{(t)}\right) & \cdots & H_{m_0n_{|N|}}\left(S_M^{(t)}\right) \\ H_{m_1n_0}\left(S_M^{(t)}\right) & H_{m_1n_1}\left(S_M^{(t)}\right) & \cdots & H_{m_1n_{|N|}}\left(S_M^{(t)}\right) \\ \vdots & \vdots & \ddots & \vdots \\ H_{m_{|M|}n_0}\left(S_M^{(t)}\right) & H_{m_{|M|}n_1}\left(S_M^{(t)}\right) & \cdots & H_{m_{|M|}n_{|N|}}\left(S_M^{(t)}\right) \end{bmatrix}.$$

According to the assumption, the purchase decision on category M product m_j sparks the purchase interest in a category N product n_j. Similar to the transition matrix $\mathbf{G}_{N|m_j}\left(S_M^{(t)},S_N^{(t)}\right)$, we need to be careful that the sparked interest happens only when a purchase decision is made. Therefore, we denote the row of $\mathbf{H}_M\left(S_M^{(t)}\right)$ corresponding to the purchase interest sparked by any product $m_j\in S_M^{(t)}$ by $\mathbf{H}_{m_j}\left(S_M^{(t)}\right)\in\left\{\mathbf{e}_{n_0},\mathbf{e}_{n_1},\ldots,\mathbf{e}_{n_{|N|}}\right\}$. When customer t's purchase interest in product n_j is sparked by his purchase decision on product m_j with probability $h_{m_jn_j}$, we have that $\mathbb{P}\left\{\mathbf{H}_{m_j}\left(S_M^{(t)}\right)=\mathbf{e}_{n_j}\right\}=h_{m_jn_j}$. When product m_j is not offered, $H_{m_jn_j}\left(S_M^{(t)}\right)$ does not exist so that we can set $\mathbb{P}\left\{\mathbf{H}_{m_j}\left(S_M^{(t)}\right)=\mathbf{0}\right\}=1\ \forall m_j\in\bar{S}_M$ for feasibility. Similarly, we still define a collection of random variables $\left\{\mathbf{H}_{m_j}\left(S_M^{(t)}\right)\mid m_j\in M_+\right\}$. Till now, we have all the random variables to construct a different likelihood function. If we have empirical data $\left\{\hat{S}_M^{(t)},\hat{S}_N^{(t)},\hat{\mathbf{Z}}_M^{(t)},\hat{\mathbf{Z}}_{N|M}^{(t)},\hat{\mathbf{F}}_M^{(t)},\hat{\mathbf{G}}_M^{(t)},\hat{\mathbf{G}}_{N|M}^{(t)},\hat{\mathbf{H}}_M^{(t)}\mid t\in T\right\}$, we have a likelihood function for customer t's path in this Markov chain choice model by

$$L_P^{(t)}(\mathbf{W})=\prod_{m_j\in M_+}\lambda_{m_j}^{\hat{F}_{m_j}^{(t)}}\prod_{m_i,m_j\in M_+}\rho_{m_im_j}^{\hat{G}_{m_im_j}^{(t)}}\prod_{\substack{m_j\in M_+\\ n_j\in N_+}}\sigma_{m_jn_j}^{\hat{H}_{m_jn_j}^{(t)}}\prod_{\substack{m_j\in M_+\\ n_i,n_j\in N_+}}\rho_{n_in_j|m_j}^{\hat{G}_{n_in_j|m_j}^{(t)}}.$$

We can consider the likelihood of the paths all the customers by $L_P(\mathbf{W})=\prod_{t\in T}L_P^{(t)}(\mathbf{W})$. Then we have the log-likelihood function for empirical data $\left\{\hat{S}_M^{(t)},\hat{S}_N^{(t)},\hat{\mathbf{Z}}_M^{(t)},\hat{\mathbf{Z}}_{N|m_j}^{(t)},\hat{\mathbf{F}}_M^{(t)},\hat{\mathbf{G}}_M^{(t)},\hat{\mathbf{G}}_{N|M}^{(t)},\hat{\mathbf{H}}_M^{(t)}\mid t\in T\right\}$ that

$$\begin{aligned} l_P^{(t)}(\mathbf{W})=&\sum_{m_j\in M_+}\hat{F}_{m_j}^{(t)}\log\lambda_{m_j}+\sum_{m_i,m_j\in M_+}\hat{G}_{m_im_j}^{(t)}\log\rho_{m_im_j}\\ &+\sum_{\substack{m_j\in M_+\\ n_j\in N_+}}\hat{H}_{m_jn_j}^{(t)}\log\sigma_{m_jn_j}+\sum_{\substack{m_j\in M_+\\ n_i,n_j\in N_+}}\hat{G}_{n_in_j|m_j}^{(t)}\log\rho_{n_in_j|m_j}. \end{aligned}$$

Similar to the maximization problem (1), the optimal feasible parameters that fit the given data maximize this log-likelihood function. Therefore, to estimate the parameters, we can formulate and solve an optimization problem that

$$\begin{aligned}
\max_{\mathbf{W}} \quad & l_P(\mathbf{W}) \\
\text{s.t.} \quad & \sum_{m_j \in M_+} \lambda_{m_j} = 1, \\
& \sum_{m_j \in M_+} \rho_{m_i m_j} = 1 \qquad \forall m_i \in M_+, \\
& \sum_{n_j \in N_+} \rho_{n_i n_j | m_j} = 1 \quad \forall m_j \in M_+,\ n_i \in N_+, \\
& \sum_{n_j \in N_+} \sigma_{m_j n_j} = 1 \qquad \forall m_j \in M_+.
\end{aligned} \tag{2}$$

However, since we do not have access to data $\left\{\hat{\mathbf{F}}_M^{(t)}, \hat{\mathbf{G}}_M^{(t)}, \hat{\mathbf{G}}_{N|M}^{(t)}, \hat{\mathbf{H}}_M^{(t)} \mid t \in T\right\}$, we can solve this problem iteratively by an expectation-maximization algorithm: we calculate the conditional expectation of these random variables with previous obtained parameters, solve the optimization problem with then estimated data, then update parameters and repeat this process until the parameters converge.

3.2 Conditional Expectation of Desired Random Variables

The expectation-maximization algorithm solves problem iteratively. In each iteration, we have an expectation step and a maximization step. We discuss the expectation step first. In the very beginning, we initialize the parameters to be $\mathbf{W}^{(1)} = \left(\mathbf{\Lambda}_M^{(1)}, \mathbf{P}_M^{(1)}, \mathbf{P}_{N|M}^{(1)}, \mathbf{S}^{(1)}\right)$ and in each iteration l, we use $\mathbf{W}^{(l)} = \left(\mathbf{\Lambda}_M^{(l)}, \mathbf{P}_M^{(l)}, \mathbf{P}_{N|M}^{(l)}, \mathbf{S}^{(l)}\right)$ to estimate the conditional expectation $\left\{\hat{\mathbf{F}}_M^{(t)}, \hat{\mathbf{G}}_M^{(t)}, \hat{\mathbf{G}}_{N|M}^{(t)}, \hat{\mathbf{H}}_M^{(t)} \mid t \in T\right\}$ by

$$\begin{aligned}
\hat{F}_{m_j}^{(t,l)} &= \mathbb{E}\left[F_{m_j} \mid \mathbf{Z}_M\left(\hat{S}_M^{(t)}\right) = \hat{\mathbf{Z}}_M^{(t)}, \mathbf{W}^{(l)}\right], \\
\hat{G}_{m_i m_j}^{(t,l)} &= \mathbb{E}\left[G_{m_i m_j}\left(\hat{S}_M^{(t)}\right) \mid \mathbf{Z}_M\left(\hat{S}_M^{(t)}\right) = \hat{\mathbf{Z}}_M^{(t)}, \mathbf{W}^{(l)}\right], \\
\hat{G}_{n_i n_j | m_j}^{(t,l)} &= \mathbb{E}\left[G_{n_i n_j | m_j}\left(\hat{S}_M^{(t)}, \hat{S}_N^{(t)}\right) \mid \mathbf{Z}_M\left(\hat{S}_M^{(t)}\right) = \hat{\mathbf{Z}}_M^{(t)}, \mathbf{Z}_{N|m_j}\left(\hat{S}_M^{(t)}, \hat{S}_N^{(t)}\right) = \hat{\mathbf{Z}}_{N|m_j}^{(t)}, \mathbf{W}^{(l)}\right], \\
\hat{H}_{m_j n_j}^{(t,l)} &= \mathbb{E}\left[H_{m_j n_j}\left(\hat{S}_M^{(t)}, \hat{S}_N^{(t)}\right) \mid \mathbf{Z}_M\left(\hat{S}_M^{(t)}\right) = \hat{\mathbf{Z}}_M^{(t)}, \mathbf{Z}_{N|m_j}\left(\hat{S}_M^{(t)}, \hat{S}_N^{(t)}\right) = \hat{\mathbf{Z}}_{N|m_j}^{(t)}, \mathbf{W}^{(l)}\right].
\end{aligned}$$

For convenience, we assume that $\mathbf{Z}_M\left(\hat{S}_M^{(t)}\right) = \hat{\mathbf{Z}}_M^{(t)} = \mathbf{e}_{m_k}$ for some $m_k \in \hat{S}_M$ and $\mathbf{Z}_{N|m_k}\left(\hat{S}_M^{(t)}, \hat{S}_N^{(t)}\right) = \hat{\mathbf{Z}}_{N|m_k}^{(t)} = \mathbf{e}_{n_k}$ for some $n_k \in \hat{S}_N$.

3.2.1 Estimation of $\left\{\hat{\mathbf{F}}_M^{(t)} \mid t \in T\right\}$

For binary variable F_{m_j}, we have $\mathbb{E}\left[F_{m_j}\right] = \mathbb{P}\left\{F_{m_j} = 1\right\}$. Therefore, we can rewrite the conditional expectation and apply the Bayes theorem to have

$$\begin{aligned}
\hat{F}_{m_j}^{(t,l)} &= \mathbb{E}\left[F_{m_j} \mid \mathbf{Z}_M\left(\hat{S}_M^{(t)}\right) = \hat{\mathbf{Z}}_M^{(t)}, \mathbf{W}^{(l)}\right] \\
&= \mathbb{P}\left\{F_{m_j} = 1 \mid \mathbf{Z}_M\left(\hat{S}_M^{(t)}\right) = \mathbf{e}_{m_k}, \mathbf{W}^{(l)}\right\} \\
&= \frac{\mathbb{P}\left\{\mathbf{Z}_M\left(\hat{S}_M^{(t)}\right) = \mathbf{e}_{m_k} \mid F_{m_j} = 1, \mathbf{W}^{(l)}\right\} \mathbb{P}\left\{F_{m_j} = 1 \mid \mathbf{W}^{(l)}\right\}}{\mathbb{P}\left\{\mathbf{Z}_M\left(\hat{S}_M^{(t)}\right) = \mathbf{e}_{m_k} \mid \mathbf{W}^{(l)}\right\}}.
\end{aligned}$$

From our model setup, we have that $\mathbb{P}\left\{\mathbf{Z}_M\left(\hat{S}_M^{(t)}\right)=\mathbf{e}_{m_k} \mid \mathbf{W}^{(l)}\right\}=\phi_{m_k}\left(\hat{S}_M^{(t)} \mid \mathbf{W}^{(l)}\right)$, which can be obtained by solving a system of linear equations. Also, by the definition of $\mathbf{F}_M$, we have $\mathbb{P}\left\{F_{m_j}=1 \mid \mathbf{W}^{(l)}\right\}=\lambda_{m_j}^{(l)}$. We can observe that when m_j is offered, customer t will purchase product m_j if he arrives into the system first visiting product m_j. Therefore, when we have assortment $\hat{S}_M$ for customer t such that $m_j \in \hat{S}_M$, we can conclude that $\mathbb{P}\left\{\mathbf{Z}_M\left(\hat{S}_M^{(t)}\right)=\mathbf{e}_{m_k} \mid F_{m_j}=1, \mathbf{W}^{(l)}\right\}=\mathbb{I}_{m_j=m_k}$, where $\mathbb{I}_{m_j=m_k}=1$ if $m_j=m_k$ and $\mathbb{I}_{m_j=m_k}=0$ if $m_j \neq m_k$. When product m_j is not offered, we can calculate this probability by solving a system of linear equations. We still use the idea that the purchase decision on product m_k can be made when the customer changes his decision from product m_i not in assortment $\hat{S}_M$ to product m_k. So we define a new variable $\theta_{m_k}\left(S_M \mid \hat{\mathbf{F}}_M^{(t)}, \mathbf{W}^{(l)}\right)=\mathbb{P}\left\{\mathbf{Z}_M\left(S_M\right)=\mathbf{e}_{m_k} \mid \mathbf{F}_M=\hat{\mathbf{F}}_M^{(t)}, \mathbf{W}^{(l)}\right\}$ for the probability that customer t visits product m_k given a previous visiting on product m_j. For any not offered product $m_k \in \bar{\hat{S}}_M^{(t)}$, we have that

$$\theta_{m_k}\left(\hat{S}_M^{(t)} \mid \mathbf{e}_{m_j}, \mathbf{W}^{(l)}\right)=\rho_{m_j m_k}+\sum_{m_i \in \bar{\hat{S}}_M^{(t)}} \rho_{m_i m_k} \theta_{m_i}\left(\hat{S}_M^{(t)} \mid \mathbf{e}_{m_j}, \mathbf{W}^{(l)}\right).$$

We use $\mathbf{\Theta}_{\bar{S}_M}\left(\hat{S}_M^{(t)} \mid \mathbf{e}_{m_j}, \mathbf{W}^{(l)}\right)=\left\{\theta_{m_i}\left(\hat{S}_M^{(t)} \mid \mathbf{e}_{m_j}, \mathbf{W}^{(l)}\right) \mid m_i, m_j \in \bar{\hat{S}}_M^{(t)}\right\}$ to denote the vector of such probabilities for not offered products in category M. We also define the transition matrix of not offered products in category M by $\mathbf{P}_{\bar{S}_M}$ and the transition vector from product m_j to the not offered products by $\mathbf{P}_{m_j \bar{S}_M}$. Now we have the solution of the system of linear equations above by $\mathbf{\Theta}_{\bar{S}_M}\left(\hat{S}_M^{(t)} \mid \mathbf{e}_{m_j}, \mathbf{W}^{(l)}\right)=\left(\mathbf{I}-\mathbf{P}_{\bar{S}_M}\right)^{-\mathrm{T}} \mathbf{P}_{m_j \bar{S}_M}$. Then we can plug the solution into $\hat{F}_{m_j}^{(t,l)}$ and have that

$$\hat{F}_{m_j}^{(t,l)}=\frac{\theta_{m_k}\left(\hat{S}_M^{(t)} \mid \mathbf{e}_{m_j}, \mathbf{W}^{(l)}\right) \lambda_{m_j}^{(l)}}{\phi_{m_k}\left(\hat{S}_M^{(t)} \mid \mathbf{W}^{(l)}\right)}.$$

3.2.2 Estimation of $\left\{\hat{\mathbf{G}}_M^{(t)} \mid t \in T\right\}$

In an infinite-horizon Markov chain, any transition from product m_i to product m_j can be expressed as an event that customer t visits product m_i in stage r and product m_j i stage $r+1$ for some $r \in \mathbb{Z}^+$. We define a binary variable $V_{m_j}^r\left(S_M^{(t)}\right)=1$ if customer t visits product m_j in stage r given assortment S_M. Otherwise, we have $V_{m_j}^r\left(S_M^{(t)}\right)=0$. Therefore, we can define a collection of random variables by $\left\{\mathbf{V}_M^r\left(S_M^{(t)}\right) \mid r \in \mathbb{Z}^+\right\}$, where

$$\mathbf{V}_M^r\left(S_M^{(t)}\right)=\left(V_{m_0}^r\left(S_M^{(t)}\right), V_{m_1}^r\left(S_M^{(t)}\right), \ldots, V_{m_{|M|}}^r\left(S_M^{(t)}\right)\right) \in\left\{\mathbf{e}_{m_0}, \mathbf{e}_{m_1}, \ldots, \mathbf{e}_{m_{|M|}}\right\}.$$

For binary variable $G_{m_i m_j}\left(\hat{S}_M^{(t)}\right)$, we have $\mathbb{E}\left[G_{m_i m_j}\left(\hat{S}_M^{(t)}\right)\right]=\sum_{r=1}^{\infty} \mathbb{P}\left\{V_{m_i}^r\left(\hat{S}_M^{(t)}\right)=V_{m_j}^{r+1}\left(\hat{S}_M^{(t)}\right)=1\right\}$. Then we can rewrite $\hat{G}_{m_i m_j}^{(t,l)}$ by

$$\begin{aligned}
\hat{G}_{m_i m_j}^{(t,l)} &=\mathbb{E}\left[G_{m_i m_j}\left(\hat{S}_M^{(t)}\right) \mid \mathbf{Z}_M\left(\hat{S}_M^{(t)}\right)=\hat{\mathbf{Z}}_M^{(t)}, \mathbf{W}^{(l)}\right] \\
&=\sum_{r=1}^{\infty} \mathbb{P}\left\{V_{m_i}^r\left(\hat{S}_M^{(t)}\right)=V_{m_j}^{r+1}\left(\hat{S}_M^{(t)}\right)=1 \mid \mathbf{Z}_M\left(\hat{S}_M^{(t)}\right)=\mathbf{e}_{m_k}, \mathbf{W}^{(l)}\right\}.
\end{aligned}$$

To further simplify the expression of $\hat{G}_{m_i m_j}^{(t,l)}$, we can put the event $V_{m_i}^r\left(\hat{S}_M^{(t)}\right)=1$ to the conditional part so that we can use the transition parameter $\mathbf{P}_M$ in this expression. We apply the definition of conditional probability and have that

$$\begin{aligned}
\hat{G}_{m_i m_j}^{(t,l)}=\sum_{r=1}^{\infty} & \mathbb{P}\left\{V_{m_j}^{r+1}\left(\hat{S}_M^{(t)}\right)=1 \mid \mathbf{Z}_M\left(\hat{S}_M^{(t)}\right)=\mathbf{e}_{m_k}, V_{m_i}^r\left(\hat{S}_M^{(t)}\right)=1, \mathbf{W}^{(l)}\right\} \\
& \times \mathbb{P}\left\{V_{m_i}^r\left(\hat{S}_M^{(t)}\right)=1 \mid \mathbf{Z}_M\left(\hat{S}_M^{(t)}\right)=\mathbf{e}_{m_k}, \mathbf{W}^{(l)}\right\}.
\end{aligned}$$

Then we use the Bayes theorem to calculate the two probabilities above respectively. For the first probability, we have that

$$\begin{aligned}
&\mathbb{P}\left\{V_{m_j}^{r+1}\left(\hat{S}_M^{(t)}\right)=1 \mid \mathbf{Z}_M\left(\hat{S}_M^{(t)}\right)=\mathbf{e}_{m_k}, V_{m_i}^r\left(\hat{S}_M^{(t)}\right)=1, \mathbf{W}^{(l)}\right\} \\
=&\frac{\mathbb{P}\left\{\mathbf{Z}_M\left(\hat{S}_M^{(t)}\right)=\mathbf{e}_{m_k} \mid V_{m_j}^{r+1}\left(\hat{S}_M^{(t)}\right)=V_{m_i}^r\left(\hat{S}_M^{(t)}\right)=1, \mathbf{W}^{(l)}\right\} \mathbb{P}\left\{V_{m_j}^{r+1}\left(\hat{S}_M^{(t)}\right)=1 \mid V_{m_i}^r\left(\hat{S}_M^{(t)}\right)=1, \mathbf{W}^{(l)}\right\}}{\mathbb{P}\left\{\mathbf{Z}_M\left(\hat{S}_M^{(t)}\right)=\mathbf{e}_{m_k} \mid V_{m_i}^r\left(\hat{S}_M^{(t)}\right)=1, \mathbf{W}^{(l)}\right\}}.
\end{aligned}$$

By the definition of purchase interest transition, we can easily see that $\mathbb{P}\left\{V_{m_j}^{r+1}\left(\hat{S}_M^{(t)}\right)=1 \mid V_{m_i}^r\left(\hat{S}_M^{(t)}\right)=1, \mathbf{W}^{(l)}\right\}=\rho_{m_i m_j}^{(l)}$. Since we are dealing with an infinite-horizon Markov chain model, that customer t purchases product m_k conditioned on a previous visit to product m_j in stage r or $r+1$ has no difference with that customer t purchases product m_k conditioned on visiting product m_j in the first stage. Therefore, we can keep using the same defined variables and solving the same system of linear equations as what we do previous section. We have that

$$\mathbb{P}\left\{V_{m_j}^{r+1}\left(\hat{S}_M^{(t)}\right)=1 \mid \mathbf{Z}_M\left(\hat{S}_M^{(t)}\right)=\mathbf{e}_{m_k}, V_{m_i}^r\left(\hat{S}_M^{(t)}\right)=1, \mathbf{W}^{(l)}\right\}=\frac{\theta_{m_k}\left(\hat{S}_M^{(t)} \mid \mathbf{e}_{m_j}, \mathbf{W}^{(l)}\right) \rho_{m_i m_j}^{(l)}}{\theta_{m_k}\left(\hat{S}_M^{(t)} \mid \mathbf{e}_{m_i}, \mathbf{W}^{(l)}\right)}.$$

Then we want to calculate $\mathbb{P}\left\{V_{m_i}^r\left(\hat{S}_M^{(t)}\right)=1 \mid \mathbf{Z}_M\left(\hat{S}_M^{(t)}\right)=\mathbf{e}_{m_k}, \mathbf{W}^{(l)}\right\}$. We still use Bayes theorem to rewrite it by

$$\begin{aligned}
&\mathbb{P}\left\{V_{m_i}^r\left(\hat{S}_M^{(t)}\right)=1 \mid \mathbf{Z}_M\left(\hat{S}_M^{(t)}\right)=\mathbf{e}_{m_k}, \mathbf{W}^{(l)}\right\} \\
=&\frac{\mathbb{P}\left\{\mathbf{Z}_M\left(\hat{S}_M^{(t)}\right)=\mathbf{e}_{m_k} \mid V_{m_i}^r\left(\hat{S}_M^{(t)}\right)=1, \mathbf{W}^{(l)}\right\} \mathbb{P}\left\{V_{m_i}^r\left(\hat{S}_M^{(t)}\right)=1 \mid \mathbf{W}^{(l)}\right\}}{\mathbb{P}\left\{\mathbf{Z}_M\left(\hat{S}_M^{(t)}\right)=\mathbf{e}_{m_k} \mid \mathbf{W}^{(l)}\right\}} \\
=&\frac{\theta_{m_k}\left(\hat{S}_M^{(t)} \mid \mathbf{e}_{m_i}, \mathbf{W}^{(l)}\right) \mathbb{P}\left\{V_{m_i}^r\left(\hat{S}_M^{(t)}\right)=1 \mid \mathbf{W}^{(l)}\right\}}{\phi_{m_k}\left(\hat{S}_M^{(t)} \mid \mathbf{W}^{(l)}\right)}.
\end{aligned}$$

Therefore, we can use these results to determine $\hat{G}_{m_i m_j}^{(t,l)}$ by

$$\begin{aligned}
\hat{G}_{m_i m_j}^{(t,l)} &= \sum_{r=1}^{\infty} \frac{\theta_{m_k}\left(\hat{S}_M^{(t)} \mid \mathbf{e}_{m_j}, \mathbf{W}^{(l)}\right) \rho_{m_i m_j}^{(l)}}{\phi_{m_k}\left(\hat{S}_M^{(t)} \mid \mathbf{W}^{(l)}\right)} \mathbb{P}\left\{V_{m_i}^r\left(\hat{S}_M^{(t)}\right)=1 \mid \mathbf{W}^{(l)}\right\} \\
&= \frac{\theta_{m_k}\left(\hat{S}_M^{(t)} \mid \mathbf{e}_{m_j}, \mathbf{W}^{(l)}\right) \rho_{m_i m_j}^{(l)}}{\phi_{m_k}\left(\hat{S}_M^{(t)} \mid \mathbf{W}^{(l)}\right)} \sum_{r=1}^{\infty} \mathbb{P}\left\{V_{m_i}^r\left(\hat{S}_M^{(t)}\right)=1 \mid \mathbf{W}^{(l)}\right\}.
\end{aligned}$$

We know that the probability that customer t visits product m_j is equal to sum of the probability that he visits product m_j in every stage. According to the definition of $V_{m_i}^r\left(\hat{S}_M^{(t)}\right)$, we know that $\sum_{r=1}^{\infty} \mathbb{P}\left\{V_{m_i}^r\left(\hat{S}_M^{(t)}\right)=1 \mid \mathbf{W}^{(l)}\right\}=\phi_{m_i}\left(\hat{S}_M^{(t)} \mid \mathbf{W}^{(l)}\right)$. In conclusion, we have the expression for $\hat{G}_{m_i m_j}^{(t,l)}$ that

$$\hat{G}_{m_i m_j}^{(t,l)}=\theta_{m_k}\left(\hat{S}_M^{(t)} \mid \mathbf{e}_{m_j}, \mathbf{W}^{(l)}\right) \rho_{m_i m_j}^{(l)} \frac{\phi_{m_i}\left(\hat{S}_M^{(t)} \mid \mathbf{W}^{(l)}\right)}{\phi_{m_k}\left(\hat{S}_M^{(t)} \mid \mathbf{W}^{(l)}\right)}.$$

3.2.3 Estimation of $\left\{\hat{\mathbf{G}}_{N|M}^{(t)} \mid t \in T\right\}$

According to our model, both the initial interests and transition interests in category N products both depends on the purchase decision of category M products. Therefore, when we estimate the transition interest within category N products, we follow the same procedure as estimating category M products, but we need to consider the conditional probability given a purchase decision $\hat{\mathbf{Z}}_M^{(t)} = \mathbf{e}_{m_k}$. Thus, we adapt the previously defined variable $\theta_{m_k}\left(S_M \mid \hat{\mathbf{F}}_M^{(t)}, \mathbf{W}^{(l)}\right)$ to be $\theta_{n_k|m_k}\left(S_M, S_N \mid \hat{\mathbf{H}}_{m_k}^{(t)}, \mathbf{W}^{(l)}\right) = \mathbb{P}\left\{\mathbf{Z}_N\left(S_M, S_N\right) = \mathbf{e}_{n_k} \mid \mathbf{H}_{m_k} = \hat{\mathbf{H}}_{m_k}^{(t)}, \mathbf{W}^{(l)}\right\}$ for the probability that customer t visits product n_k given a previous visiting on product n_j and a purchase decision on product m_k. For any not offered product $n_i \in \bar{\hat{S}}_N^{(t)}$, we have that

$$\theta_{n_k|m_k}\left(\hat{S}_M^{(t)}, \hat{S}_N^{(t)} \mid \mathbf{e}_{n_j}, \mathbf{W}^{(l)}\right) = \rho_{n_j n_k|m_k} + \sum_{n_i \in \bar{\hat{S}}_N^{(t)}} \rho_{n_i n_k|m_k} \theta_{n_i|m_k}\left(\hat{S}_M^{(t)}, \hat{S}_N^{(t)} \mid \mathbf{e}_{n_j}, \mathbf{W}^{(l)}\right).$$

Similar to the previous sections, we have that $\mathbf{\Theta}_{\bar{S}_{N|m_k}}\left(\hat{S}_M^{(t)}, \hat{S}_N^{(t)} \mid \mathbf{e}_{m_j}, \mathbf{W}^{(l)}\right) = \left(\mathbf{I} - \mathbf{P}_{\bar{S}_{N|m_k}}\right)^{-\mathrm{T}} \mathbf{P}_{n_j \bar{S}_N|m_k}$, where $\mathbf{P}_{n_j \bar{S}_N|m_k}$ is the row of transition matrix $\mathbf{P}_{N|m_k}$ from product n_j to all the not offered products, is the solution for the system of linear equations above. Also, we use the conditional probability of visiting product n_j given the purchase decision on m_k instead of the unconditional probability. Then we just use the same method but replace all the unconditional probability by the corresponding conditional version and calculate $\hat{G}_{n_i n_j|m_k}^{(t,l)}$ by

$$\hat{G}_{n_i n_j|m_k}^{(t,l)} = \theta_{n_k|m_k}\left(\hat{S}_M^{(t)}, \hat{S}_N^{(t)} \mid \mathbf{e}_{n_j}, \mathbf{W}^{(l)}\right) \rho_{n_i n_j|m_k}^{(l)} \frac{\phi_{n_i|m_k}\left(\hat{S}_M^{(t)}, \hat{S}_N^{(t)} \mid \mathbf{W}^{(l)}\right)}{\phi_{n_k|m_k}\left(\hat{S}_M^{(t)}, \hat{S}_N^{(t)} \mid \mathbf{W}^{(l)}\right)}.$$

Last but not least, it is very natural to set $\hat{\mathbf{G}}_{N|m_{k'}} = \mathbf{0}$ for any $k' \neq k$.

3.2.4 Estimation of $\left\{\hat{\mathbf{H}}_M^{(t)} \mid t \in T\right\}$

Now we need to consider the conditional expectation of customer t's interest in category N products sparked by product m_j given that he eventually purchases product m_k and product n_k. Similar to the previous section, we still define a binary variable $V_{n_j}^r\left(S_M^{(t)}, S_N^{(t)}\right) = 1$ if customer t visits product n_j in stage r given assortment S_M and S_N. Otherwise, $V_{n_j}^r\left(S_M^{(t)}, S_N^{(t)}\right) = 0$. This time we need to consider that customer t visits product m_j in stage r and product n_j in stage $r+1$. We can see that stage r is the last stage customer t visits category M products. So, it is infeasible when product m_j is not the purchase decision of customer t among category M products. We can set that $\hat{H}_{m_j n_j}^{(t)} = 0\ \forall m_j \neq m_k$ and only calculate $\hat{H}_{m_k n_j}^{(t,l)}$. For binary variable $H_{m_k n_j}\left(\hat{S}_M^{(t)}, \hat{S}_N^{(t)}\right)$, we have $\mathbb{E}\left[H_{m_k n_j}\left(\hat{S}_M^{(t)}, \hat{S}_N^{(t)}\right)\right] = \sum_{r=1}^{\infty} \mathbb{P}\left\{V_{m_k}^r\left(\hat{S}_M^{(t)}\right) = V_{n_j}^{r+1}\left(\hat{S}_M^{(t)}, \hat{S}_N^{(t)}\right) = 1\right\}$. Then we can rewrite $\hat{H}_{m_j n_j}^{(t,l)}$ by

$$\begin{aligned}
\hat{H}_{m_k n_j}^{(t,l)} &= \mathbb{E}\left[H_{m_k n_j}\left(\hat{S}_M^{(t)}, \hat{S}_N^{(t)}\right) \mid \mathbf{Z}_M\left(\hat{S}_M^{(t)}\right) = \hat{\mathbf{Z}}_M^{(t)}, \mathbf{Z}_{N|m_k}\left(\hat{S}_M^{(t)}, \hat{S}_N^{(t)}\right) = \hat{\mathbf{Z}}_{N|m_k}^{(t)}, \mathbf{W}^{(l)}\right] \\
&= \sum_{r=1}^{\infty} \mathbb{P}\left\{V_{m_k}^r\left(\hat{S}_M^{(t)}\right) = V_{n_j}^{r+1}\left(\hat{S}_M^{(t)}, \hat{S}_N^{(t)}\right) = 1 \mid \mathbf{Z}_M\left(\hat{S}_M^{(t)}\right) = \mathbf{e}_{m_k}, \mathbf{Z}_{N|m_k}\left(\hat{S}_M^{(t)}, \hat{S}_N^{(t)}\right) = \mathbf{e}_{n_k}, \mathbf{W}^{(l)}\right\}.
\end{aligned}$$

We follow the same idea to move the event $V_{m_k}^r\left(\hat{S}_M^{(t)}\right) = 1$ to the conditional part and use the definition of conditional probability to rewrite $\hat{H}_{m_k n_j}^{(t,l)}$ by

$$\begin{aligned}
\hat{H}_{m_k n_j}^{(t,l)} = \sum_{r=1}^{\infty} &\mathbb{P}\left\{V_{n_j}^{r+1}\left(\hat{S}_M^{(t)}, \hat{S}_N^{(t)}\right) = 1 \mid \mathbf{Z}_M\left(\hat{S}_M^{(t)}\right) = \mathbf{e}_{m_k}, \mathbf{Z}_{N|m_k}\left(\hat{S}_M^{(t)}, \hat{S}_N^{(t)}\right) = \mathbf{e}_{n_k}, V_{m_k}^r\left(\hat{S}_M^{(t)}\right) = 1, \mathbf{W}^{(l)}\right\} \\
&\times \mathbb{P}\left\{V_{m_k}^r\left(\hat{S}_M^{(t)}\right) = 1 \mid \mathbf{Z}_M\left(\hat{S}_M^{(t)}\right) = \mathbf{e}_{m_k}, \mathbf{Z}_{N|m_k}\left(\hat{S}_M^{(t)}, \hat{S}_N^{(t)}\right) = \mathbf{e}_{n_k}, \mathbf{W}^{(l)}\right\}.
\end{aligned}$$

We calculate both of the two probabilities in the formula above by Bayes theorem. We rewrite the first probability by

$$\begin{aligned}
&\mathbb{P}\left\{V_{n_j}^{r+1}\left(\hat{S}_M^{(t)},\hat{S}_N^{(t)}\right)=1 \mid \mathbf{Z}_M\left(\hat{S}_M^{(t)}\right)=\mathbf{e}_{m_k},\mathbf{Z}_{N|m_k}\left(\hat{S}_M^{(t)},\hat{S}_N^{(t)}\right)=\mathbf{e}_{n_k},V_{m_k}^r\left(\hat{S}_M^{(t)}\right)=1,\mathbf{W}^{(l)}\right\}\\
=&\mathbb{P}\left\{\mathbf{Z}_M\left(\hat{S}_M^{(t)}\right)=\mathbf{e}_{m_k},\mathbf{Z}_{N|m_k}\left(\hat{S}_M^{(t)},\hat{S}_N^{(t)}\right)=\mathbf{e}_{n_k} \mid V_{m_k}^r\left(\hat{S}_M^{(t)}\right)=1,V_{n_j}^{r+1}\left(\hat{S}_M^{(t)},\hat{S}_N^{(t)}\right)=1,\mathbf{W}^{(l)}\right\}\\
&\times\frac{\mathbb{P}\left\{V_{n_j}^{r+1}\left(\hat{S}_M^{(t)},\hat{S}_N^{(t)}\right)=1 \mid V_{m_k}^r\left(\hat{S}_M^{(t)}\right)=1,\mathbf{W}^{(l)}\right\}}{\mathbb{P}\left\{\mathbf{Z}_M\left(\hat{S}_M^{(t)}\right)=\mathbf{e}_{m_k},\mathbf{Z}_{N|m_k}\left(\hat{S}_M^{(t)},\hat{S}_N^{(t)}\right)=\mathbf{e}_{n_k} \mid V_{m_k}^r\left(\hat{S}_M^{(t)}\right)=1,\mathbf{W}^{(l)}\right\}}.
\end{aligned}$$

When we are given $V_{m_k}^r\left(\hat{S}_M^{(t)}\right)=1$, we know that customer t visits product m_k and m_k is offered according to the assumption. Therefore, the customer must purchase product m_k and the event $\mathbf{Z}_M\left(\hat{S}_M^{(t)}\right)=\mathbf{e}_{m_k}$ must happen given $V_{m_k}^r\left(\hat{S}_M^{(t)}\right)=1$. So, we only need to consider $\mathbb{P}\left\{\mathbf{Z}_{N|m_k}\left(\hat{S}_M^{(t)},\hat{S}_N^{(t)}\right)=\mathbf{e}_{n_k} \mid V_{n_j}^{r+1}\left(\hat{S}_M^{(t)},\hat{S}_N^{(t)}\right)=1,\mathbf{W}^{(l)}\right\}$, which can be obtained by solving a system of linear equations as is demonstrated in the previous section. We keep the same notation for that by $\theta_{n_k|m_k}\left(\hat{S}_M^{(t)},\hat{S}_N^{(t)} \mid \mathbf{e}_{n_j},\mathbf{W}^{(l)}\right)$. By the previous definition of interest spark, we have that $\mathbb{P}\left\{V_{n_j}^{r+1}\left(\hat{S}_M^{(t)},\hat{S}_N^{(t)}\right)=1 \mid V_{m_k}^r\left(\hat{S}_M^{(t)}\right)=1,\mathbf{W}^{(l)}\right\}=\sigma_{m_k n_j}^{(l)}$. Also, from our previous definition of probability of visiting product n_j conditioned on purchase decision on product m_j under assortment S_M and S_N, we have that $\mathbb{P}\left\{\mathbf{Z}_M\left(\hat{S}_M^{(t)}\right)=\mathbf{e}_{m_k},\mathbf{Z}_{N|m_k}\left(\hat{S}_M^{(t)},\hat{S}_N^{(t)}\right)=\mathbf{e}_{n_k} \mid V_{m_k}^r\left(\hat{S}_M^{(t)}\right)=1,\mathbf{W}^{(l)}\right\}=\phi_{n_k|m_k}\left(\hat{S}_M^{(t)},\hat{S}_N^{(t)}\right)$. We combine all these results and have that

$$\begin{aligned}
&\mathbb{P}\left\{V_{n_j}^{r+1}\left(\hat{S}_M^{(t)},\hat{S}_N^{(t)}\right)=1 \mid \mathbf{Z}_M\left(\hat{S}_M^{(t)}\right)=\mathbf{e}_{m_k},\mathbf{Z}_{N|m_k}\left(\hat{S}_M^{(t)},\hat{S}_N^{(t)}\right)=\mathbf{e}_{n_k},V_{m_k}^r\left(\hat{S}_M^{(t)}\right)=1,\mathbf{W}^{(l)}\right\}\\
=&\frac{\theta_{n_k|m_k}\left(\hat{S}_M^{(t)},\hat{S}_N^{(t)} \mid \mathbf{e}_{n_j},\mathbf{W}^{(l)}\right)\sigma_{m_k n_j}^{(l)}}{\phi_{n_k|m_k}\left(\hat{S}_M^{(t)},\hat{S}_N^{(t)}\right)}.
\end{aligned}$$

Then we deal with $\mathbb{P}\left\{V_{m_k}^r\left(\hat{S}_M^{(t)}\right)=1 \mid \mathbf{Z}_M\left(\hat{S}_M^{(t)}\right)=\mathbf{e}_{m_k},\mathbf{Z}_{N|m_k}\left(\hat{S}_M^{(t)},\hat{S}_N^{(t)}\right)=\mathbf{e}_{n_k},\mathbf{W}^{(l)}\right\}$. Similar to the previous procedures, we apply Bayes theorem to rewrite this expression by

$$\begin{aligned}
&\mathbb{P}\left\{V_{m_k}^r\left(\hat{S}_M^{(t)}\right)=1 \mid \mathbf{Z}_M\left(\hat{S}_M^{(t)}\right)=\mathbf{e}_{m_k},\mathbf{Z}_{N|m_k}\left(\hat{S}_M^{(t)},\hat{S}_N^{(t)}\right)=\mathbf{e}_{n_k},\mathbf{W}^{(l)}\right\}\\
=&\frac{\mathbb{P}\left\{\mathbf{Z}_{N|m_k}\left(\hat{S}_M^{(t)},\hat{S}_N^{(t)}\right)=\mathbf{e}_{n_k} \mid V_{m_k}^r\left(\hat{S}_M^{(t)}\right)=1,\mathbf{W}^{(l)}\right\}\mathbb{P}\left\{V_{m_k}^r\left(\hat{S}_M^{(t)}\right)=1 \mid \mathbf{W}^{(l)}\right\}}{\mathbb{P}\left\{\mathbf{Z}_M\left(\hat{S}_M^{(t)}\right)=\mathbf{e}_{m_k},\mathbf{Z}_{N|m_k}\left(\hat{S}_M^{(t)},\hat{S}_N^{(t)}\right)=\mathbf{e}_{n_k} \mid \mathbf{W}^{(l)}\right\}}.
\end{aligned}$$

From the model setup, we denote the joint probability of purchasing product m_j and n_j by $\phi_{m_i n_j}\left(S_M,S_N\right)=\phi_{m_j}\left(S_M\right)\phi_{n_j|m_j}\left(S_M,S_N\right)$. Therefore, this probability above can be calculated by

$$\begin{aligned}
&\mathbb{P}\left\{V_{m_k}^r\left(\hat{S}_M^{(t)}\right)=1 \mid \mathbf{Z}_M\left(\hat{S}_M^{(t)}\right)=\mathbf{e}_{m_k},\mathbf{Z}_{N|m_k}\left(\hat{S}_M^{(t)},\hat{S}_N^{(t)}\right)=\mathbf{e}_{n_k},\mathbf{W}^{(l)}\right\}\\
=&\frac{\phi_{n_k|m_k}\left(\hat{S}_M^{(t)},\hat{S}_N^{(t)}\right)\mathbb{P}\left\{V_{m_k}^r\left(\hat{S}_M^{(t)}\right)=1 \mid \mathbf{W}^{(l)}\right\}}{\phi_{m_k}\left(\hat{S}_M^{(t)}\right)\phi_{n_k|m_k}\left(\hat{S}_M^{(t)},\hat{S}_N^{(t)}\right)}=\frac{\mathbb{P}\left\{V_{m_k}^r\left(\hat{S}_M^{(t)}\right)=1 \mid \mathbf{W}^{(l)}\right\}}{\phi_{m_k}\left(\hat{S}_M^{(t)}\right)}.
\end{aligned}$$

The last step is to combine all these results and obtain a simple expression for $\hat{H}^{(t,l)}_{m_k n_j}$ by

$$\begin{aligned}\hat{H}^{(t,l)}_{m_k n_j} &= \sum_{r=1}^{\infty} \frac{\theta_{n_k|m_k}\left(\hat{S}^{(t)}_M, \hat{S}^{(t)}_N \mid \mathbf{e}_{n_j}, \mathbf{W}^{(l)}\right)\sigma^{(l)}_{m_k n_j}}{\phi_{m_k}\left(\hat{S}^{(t)}_M\right)\phi_{n_k|m_k}\left(\hat{S}^{(t)}_M, \hat{S}^{(t)}_N\right)} \mathbb{P}\left\{V^r_{m_k}\left(\hat{S}^{(t)}_M\right) = 1 \mid \mathbf{W}^{(l)}\right\} \\ &= \frac{\theta_{n_k|m_k}\left(\hat{S}^{(t)}_M, \hat{S}^{(t)}_N \mid \mathbf{e}_{n_j}, \mathbf{W}^{(l)}\right)\sigma^{(l)}_{m_k n_j}}{\phi_{m_k}\left(\hat{S}^{(t)}_M\right)\phi_{n_k|m_k}\left(\hat{S}^{(t)}_M, \hat{S}^{(t)}_N\right)} \sum_{r=1}^{\infty} \mathbb{P}\left\{V^r_{m_k}\left(\hat{S}^{(t)}_M\right) = 1 \mid \mathbf{W}^{(l)}\right\} \\ &= \frac{\theta_{n_k|m_k}\left(\hat{S}^{(t)}_M, \hat{S}^{(t)}_N \mid \mathbf{e}_{n_j}, \mathbf{W}^{(l)}\right)\sigma^{(l)}_{m_k n_j}}{\phi_{m_k}\left(\hat{S}^{(t)}_M\right)\phi_{n_k|m_k}\left(\hat{S}^{(t)}_M, \hat{S}^{(t)}_N\right)} \phi_{m_k}\left(\hat{S}^{(t)}_M\right) = \frac{\theta_{n_k|m_k}\left(\hat{S}^{(t)}_M, \hat{S}^{(t)}_N \mid \mathbf{e}_{n_j}, \mathbf{W}^{(l)}\right)\sigma^{(l)}_{m_k n_j}}{\phi_{n_k|m_k}\left(\hat{S}^{(t)}_M, \hat{S}^{(t)}_N\right)}.\end{aligned}$$

Now we finish all the calculations of conditional expectation of desired variables. We will solve the maximization problem with the estimated data, which will be discussed in the next section.

3.3 Update of Parameters

If we observe the structure of the maximization problem (2), it can be easily decomposed by the decision variables $\mathbf{W} = (\mathbf{\Lambda}_M, \mathbf{P}_M, \mathbf{P}_N, \mathbf{S})$. We update $\mathbf{\Lambda}_M$ first by solving the decomposed problem

$$\begin{aligned}\max_{\mathbf{\Lambda}_M} \quad & \sum_{t\in T}\sum_{m_j\in M_+} \hat{F}^{(t,l)}_{m_j} \log \lambda_{m_j} \\ \text{s.t.} \quad & \sum_{m_j \in M_+} \lambda_{m_j} = 1.\end{aligned} \tag{3}$$

Proved by Simsek and Topaloglu (2018), This maximization problem has a very simple form and a neat solution given by

$$\lambda^{(l+1)}_{m_j} = \frac{\sum_{t\in T} \hat{F}^{(t,l)}_{m_j}}{\sum_{t\in T}\sum_{m_k\in M_+} \hat{F}^{(t,l)}_{m_k}}, \ \forall m_j \in M_+.$$

Similarly, we can solve the other three decomposed maximization problems and update the parameters by

$$\begin{aligned}\rho^{(l+1)}_{m_i m_j} &= \frac{\sum_{t\in T} \hat{G}^{(t,l)}_{m_i m_j}}{\sum_{t\in T}\sum_{m_k\in M_+} \hat{G}^{(t,l)}_{m_i m_k}} \\ \rho^{(l+1)}_{n_i n_j|m_j} &= \frac{\sum_{t\in T} \hat{G}^{(t,l)}_{n_i n_j|m_j}}{\sum_{t\in T}\sum_{n_k\in N_+} \hat{G}^{(t,l)}_{n_i n_k|m_j}} \\ \sigma^{(l+1)}_{m_j n_j} &= \frac{\sum_{t\in T} \hat{H}^{(t,l)}_{m_j n_j}}{\sum_{t\in T}\sum_{n_k\in N_+} \hat{H}^{(t,l)}_{m_j n_k}}, \ \forall m_i, m_j \in M_+, \ \forall n_i, n_j \in N_+.\end{aligned}$$

3.4 Convergence of the Proposed Expectation-Maximization Algorithm

We want to show that the sequence of parameters calculated from each iteration $\left\{\mathbf{W}^{(l)} \mid l \in \mathbb{Z}^+\right\}$ converge to a local maximum of the likelihood function $L(\mathbf{W})$. We firstly define the space of parameters $\mathbf{W} = (\mathbf{\Lambda}_M, \mathbf{P}_M, \mathbf{P}_N, \mathbf{S})$ by

$$\mathbf{\Omega} = \left\{\mathbf{W} \mid \sum_{m_j\in M_+} \lambda_{m_j} = 1, \sum_{m_j\in M_+} \rho_{m_i m_j} = 1, \sum_{n_j \in N_+} \sigma_{m_j n_j} = 1, \sum_{n_j\in N_+} \rho_{n_i n_j|m_j} = 1, \ \forall m_i, m_j \in M_+, \ n_i \in N_+\right\}.$$

Then we can define the set of local maximum by $\mathbf{\Omega}^* = \left\{\mathbf{W}^* \mid \lim_{\gamma\to 0^+} \frac{L((1-\gamma)\mathbf{W}^* + \gamma\mathbf{W}) - L(\mathbf{W}^*)}{\gamma} \leq 0, \ \forall \mathbf{W} \in \mathbf{\Omega}\right\}$. The proof of convergence follows the procedure by Simsek and Topaloglu (2018), but we take the additional parameter

$\mathbf{S}$ into account. Firstly, we need to show that $L\left(\mathbf{W}^{(l+1)}\right) \geq L\left(\mathbf{W}^{(l)}\right)\ \forall l \in \mathbb{Z}^{+}$ so that we know bounded monotone sequence must converges. Moreover, we need to show that $\lim_{l\to\infty} \mathbf{W}^{(l)} = \mathbf{W}^{*} \in \mathbf{\Omega}^{*}$ and $\lim_{l\to\infty} L\left(\mathbf{W}^{(l)}\right) = L\left(\mathbf{W}^{*}\right)$. These can be guaranteed by satisfying the regularity conditions in Nettleton (1999).

3.4.1 Regularity Condition 1

The first regularity condition is that the likelihood function $L(\mathbf{W})$ is continuous and differentiable on $\mathbf{W}$ over $\mathbf{\Omega}$. According to Puterman (1994, corollary C.4), the solution of the system of linear equations $\mathbf{\Phi}_{\bar{S}_M}(S_M) = \left(\mathbf{I}-\mathbf{P}_{\bar{S}_M}\right)^{-\mathrm{T}} \mathbf{\Lambda}_{\bar{S}_M}$ always exists. Therefore, the probability of visiting not offered products $\mathbf{\Phi}_{\bar{S}_M}(S_M)$ is continuous and differentiable on all the entries involved in this operation. For the offered products, we know that $\phi_{m_j}(S_M) = \lambda_{m_j} + \sum_{m_i \in \bar{S}_M} \rho_{m_i m_j} \phi_{m_i}(S_M)\ \forall m_j \in S_M$. Therefore, we can see that $\mathbf{\Phi}_M(S_M)$ is continuous and differentiable on $(\mathbf{\Lambda}_M, \mathbf{P}_M)$. Similarly, $\mathbf{\Phi}_N(S_M, S_N)$ is continuous and differentiable on $\mathbf{W}$. The likelihood function is the product of entries of $\mathbf{\Phi}_M(S_M)$ and $\mathbf{\Phi}_N(S_M, S_N)$ so it is continuous and differentiable on $\mathbf{W}$ over $\mathbf{\Omega}$.

3.4.2 Regularity Condition 2

The second regularity condition is that $\mathbf{\Omega}_\alpha = \{\mathbf{W} \in \mathbf{\Omega} \mid L(\mathbf{W}) \geq \alpha\}$ is compact $\forall \alpha \in \mathbb{R}$. According to Heine–Borel theorem, a compact set is equivalent to a closed and bounded set over $\mathbb{R}$. By our definition, $\mathbf{\Omega}$ is closed and bounded so $\mathbf{\Omega}_\alpha \subseteq \mathbf{\Omega}$ is bounded. We can prove that $\mathbf{\Omega}_\alpha$ is closed by contradiction. Assume that a sequence defined by $\left\{\mathbf{W}^{(l)} \in \mathbf{\Omega}_\alpha \mid l \in \mathbb{Z}^{+}\right\}$ has limit $\mathbf{W}_* \notin \mathbf{\Omega}_\alpha$. Since $\mathbf{\Omega}$ is closed and bounded and $\mathbf{\Omega}_\alpha \subseteq \mathbf{\Omega}$, we have that $\mathbf{W}_* \in \mathbf{\Omega} \backslash \mathbf{\Omega}_\alpha$. We can find some $\delta > 0$ such that $L(\mathbf{W}_*) \leq \alpha - \delta < \alpha$. Since $L(\mathbf{W})$ is continuous, we can always find $\mathbf{W}^{(l)}$ such that $\left|L(\mathbf{W}_*) - L\left(\mathbf{W}^{(l)}\right)\right| < \delta$. Therefore, we have that $L\left(\mathbf{W}^{(l)}\right) < \alpha$ and $\mathbf{W}^{(l)} \notin \mathbf{\Omega}_\alpha$, which is a contradiction. Thus, any sequence $\left\{\mathbf{W}^{(l)} \in \mathbf{\Omega}_\alpha \mid l \in \mathbb{Z}^{+}\right\}$ has limit $\mathbf{W}_* \in \mathbf{\Omega}_\alpha$ so $\mathbf{\Omega}_\alpha$ is compact.

3.4.3 Regularity Condition 3

The third regularity conditon is that the path likelihood function $L_P^{(l)}(\mathbf{W})$ is continuous on $\mathbf{W}^{(l)}$ and $\mathbf{W}$ over $\mathbf{\Omega} \times \mathbf{\Omega}$. We need to show that all the entries of conditional expectation $\hat{\mathbf{F}}_M, \hat{\mathbf{G}}_M, \hat{\mathbf{G}}_N, \hat{\mathbf{H}}_M$ is continuous on $\mathbf{W}^{(l)}$. We show this continuity by the similar method for regularity condition 1. Since all the solution

$$
\begin{aligned}
\mathbf{\Phi}_{\bar{S}_M}^{(l)}\left(S_M \mid \mathbf{W}^{(l)}\right) &= \left(\mathbf{I} - \mathbf{P}_{\bar{S}_M}^{(l)}\right)^{-\mathrm{T}} \mathbf{\Lambda}_{\bar{S}_M}^{(l)} \\
\mathbf{\Phi}_{\bar{S}_N|M}^{(l)}\left(S_M, S_N \mid \mathbf{W}^{(l)}\right) &= \left(\mathbf{I} - \mathbf{P}_{\bar{S}_N}^{(l)}\right)^{-\mathrm{T}} \left(\mathbf{S}_{M\bar{S}_N}^{(l)}\right)^{\mathrm{T}} \\
\mathbf{\Theta}_{\bar{S}_M}^{(l)}\left(S_M \mid \mathbf{e}_{m_j}, \mathbf{W}^{(l)}\right) &= \left(\mathbf{I} - \mathbf{P}_{\bar{S}_M}\right)^{-\mathrm{T}} \mathbf{P}_{m_j \bar{S}_M} \\
\mathbf{\Theta}_{\bar{S}_N|M}^{(l)}\left(S_M, S_N \mid \mathbf{e}_{m_j}, \mathbf{W}^{(l)}\right) &= \left(\mathbf{I} - \mathbf{P}_{\bar{S}_N}\right)^{-\mathrm{T}} \mathbf{P}_{n_j \bar{S}_N|M}
\end{aligned}
$$

of systems of linear equations in each expectation-maximization iteration always exists, we have that all the entries of $\mathbf{\Phi}_{\bar{S}_M}^{(l)}\left(S_M \mid \mathbf{W}^{(l)}\right)$, $\mathbf{\Phi}_{\bar{S}_N|M}^{(l)}\left(S_M, S_N \mid \mathbf{W}^{(l)}\right)$, $\mathbf{\Theta}_M^{(l)}\left(S_M^{(t)} \mid \mathbf{e}_{m_j}, \mathbf{W}^{(l)}\right)$ and $\mathbf{\Theta}_{\bar{S}_N|M}^{(l)}\left(S_M, S_N \mid \mathbf{e}_{m_j}, \mathbf{W}^{(l)}\right)$ are continuous on $\mathbf{W}^{(l)}$. Since all the entries of conditional expectation $\hat{\mathbf{F}}_M, \hat{\mathbf{G}}_M, \hat{\mathbf{G}}_{N|M}, \hat{\mathbf{H}}_M$ is calculated by multiplication and division among the entries listed above and $\mathbf{W}^{(l)}$, they are continuous on $\mathbf{W}^{(l)}$. Therefore, we can conclude that $L_P^{(l)}(\mathbf{W})$ is continuous on $\mathbf{W}^{(l)}$ and $\mathbf{W}$ over $\mathbf{\Omega} \times \mathbf{\Omega}$.

4 Assortment Optimization

4.1 Linear Programming Formulation

After we have the estimation of all the parameters, we want to solve the assortment optimization problem to maximize the expected revenue. Denote the price of product m_j and n_j by r_{m_j} and r_{n_j} respectively. Firstly, we want to discuss the expected revenue g_{m_j} and g_{n_j} by cases.

- When $n_j \in S_N$, the expected revenue $g_{n_j|m_j} = r_{n_j}$.
- When $n_j \in \bar{S}_N$, the expected revenue $g_{n_j|m_j} = \sum_{n_i \in N_+} \rho_{n_j n_i|m_j} g_{n_i|m_j}$.
- When $m_j \in S_M$, the expected revenue $g_{m_j} = r_{m_j} + \sum_{n_i \in N_+} \sigma_{m_j n_i} g_{n_i|m_j}$.
- When $m_j \in \bar{S}_M$, the expected revenue $g_{m_j} = \sum_{m_i \in M_+} \rho_{m_j m_i} g_{m_i}$.

Therefore, this assortment optimization problem can be formulated by

$$
\begin{aligned}
\min \quad & \sum_{m_i \in M_+} \lambda_{m_i} g_{m_i} \\
\text{s.t.} \quad & g_{n_j|m_j} \geq r_{n_j} && \forall n_j \in N_+, \\
& g_{n_j|m_j} \geq \sum_{n_i \in N_+} \rho_{n_j n_i|m_j} g_{n_i} && \forall n_j \in N_+, \\
& g_{m_j} \geq r_{m_j} + \sum_{n_i \in N_+} \sigma_{m_j n_i} g_{n_i|m_j} && \forall m_j \in M_+, \\
& g_{m_j} \geq \sum_{m_i \in M_+} \rho_{m_j m_i} g_{m_i} && \forall m_j \in M_+.
\end{aligned}
\tag{4}
$$

This problem is a linear program with decision variables $\{g_{m_j} \mid m_j \in M_+\}$ and $\{g_{n_j} \mid n_j \in N_+\}$. We make the decision whether we offer product m_j and n_j according to which constraint is tight. When $g^*_{n_j} \geq r_{n_j}$ is tight, it is implied that $r_{n_j} \geq \sum_{n_i \in N} \rho_{n_j n_i} g^*_{n_i}$. We can receive more revenue by selling product n_j than allowing the interest transferred to other products. Then we decide to offer product n_j. On the contrary, when $g^*_{n_j} \geq \sum_{n_i \in N} \rho_{n_j n_i} g^*_{n_i}$ is tight, we receive more revenue by allowing the interest transferred from product n_j to other products than selling it. Then we decide not to offer product n_j. Thus, this assortment optimization problem can be solved efficiently. Since we know that a linear program has zero duality gap, we can easily write the dual program of the previous linear program. The dual program is more intuitive and can be interpreted by maximizing the revenue. The dual program is given by

$$
\begin{aligned}
\max \quad & \sum_{m_i \in M_+} r_{m_i} w_{m_i} + \sum_{n_i \in N_+} r_{n_i} y_{n_i} \\
\text{s.t.} \quad & w_{m_j} + x_{m_j} - \sum_{m_i \in M_+} \rho_{m_i m_j} x_i = \lambda_{m_j} && \forall m_j \in M_+, \\
& \sum_{m_i \in M_+} \sigma_{m_i n_j} w_{m_i} + y_{n_j} + z_{n_j} - \sum_{n_i \in N_+} \rho_{n_i n_j} z_i = 0 && \forall n_j \in N_+, \\
& w_{m_j},\ x_{m_j},\ y_{n_j},\ z_{n_j} \geq 0 && \forall m_j \in M_+,\ n_j \in N_+.
\end{aligned}
\tag{5}
$$

4.2 Proof of Optimality

We denote the optimal solution to the linear program above by $\mathbf{g}^*_M$ and $\mathbf{g}^*_{N|M}$. For each optimal expected revenue $g^*_{n_j|m_j}$ from product n_j conditioned on a purchase decision on product m_j, one of the first and the second constraint must be tight, otherwise we can decrease $g^*_{n_j|m_j}$ by a small ϵ and keep the solution feasible. Then the objective value will decrease as well so the $g^*_{n_j|m_j}$ is no longer the optimal solution. Similarly, we have that for each optimal expected revenue $g^*_{m_j}$ from product m_j, one of the third and the fourth constraint must be tight. Therefore, we can observe that this optimal solution satisfy that

$$
g^*_{n_j|m_j} = \max\left\{ r_{n_j}, \sum_{n_i \in N} \rho_{n_j n_i|m_j} g^*_{n_i|m_j} \right\} \text{ and } g^*_{m_j} = \max\left\{ r_{m_j} + \sum_{n_i \in N} \sigma_{m_j n_i} g^*_{n_i|m_j}, \sum_{m_i \in M} \rho_{m_j m_i} g^*_{m_i} \right\}.
$$

Then we can obtain the maximum revenue from each customer by plugging $\mathbf{g}_M^*$ and $\mathbf{g}_{N|M}^*$ into $\sum_{m_i \in M_+} \lambda_{m_i} g_{m_i}$. We can choose the optimal assortment as described in the previous subsection such that the revenue obtained by each product is equal to $\mathbf{g}_M^*$ and $\mathbf{g}_{N|M}^*$.This completes the proof of optimality.

5 Numerical Experiments

To test the performance of the proposed model, we run some numerical experiments. The data for training and testing in these experiments are generated according to the maximum utility model provided by Ghoniem et al., (2016), which also takes the asymmetric cross-selling effect into consideration. In this model, customers are classified into different segments. Each segment of customers has different reservation prices for each product. Moreover, the correlation between the two categories is modeled that when a customer purchases a product from the primary category, he will have new reservation prices for the secondary category products. The customer will choose the product with the highest price below the reservation price to maximize utility. For the generated training data, we estimate the parameters for our proposed multi-category Markov chain choice model (Model 2) by the derived expectation-maximization algorithm. In some business settings, such as most of the e-commerce, we are allowed to offer different secondary products right after the purchase decision on primary category products is made. On the contrary, for the retailers such as supermarkets, we are not allowed to offer different secondary products immediately. In the following experiments, we have both online and offline settings. Then we can find the optimal assortment and use this assortment to calculate the revenue from test data. Then we do the same thing to the benchmark model (Model 1), which is the Markov chain choice model treating different categories independently, and make comparison between the proposed model and benchmark model. The results of experiments are presented in the following table.

Table 1: Performance of Two Models

	Offline 1	Online 2	Online 3	Offline 4	Offline 5
Number of observations	10^6	10^6	2×10^6	2×10^6	2×10^6
Number of assortments	5000	5000	5000	4000	4000
Number of segments	100	100	500	1000	1000
Size of each category	10	10	10	10	10
Category 1 price	Normal	Normal	Normal	Gamma	Gamma
Reservation price	Normal	Normal	Normal	Gamma	Gamma
Model 1 accuracy	78.827%	77.481%	78.798%	31.933%	34.528%
Model 2 accuracy	78.825%	77.503%	78.756%	31.763%	34.452%
Model 1 revenue	67747	73172	74291	410560	320789
Model 2 revenue	67985	78503	77737	413795	324477

The accuracy in the table above is calculated by the total number of wrong count of the estimated expected purchases of each product compared with the actual purchases. From this table, we can see that the two models give the similar estimation accuracy. Both of the two models fit the normal distributed price cases very well, but they fit the gamma distributed price cases poorly. Furthermore, we can observe that the proposed model does improve the revenue.

6 Conclusion

We explored the assortment optimization for multi-category products. We extended the Markov chain choice model by adding the categorical transitions into the Markov chain, adapted the expectation-maximization algorithm to fit my proposed model from sales data, and solved a linear program for assortment optimization. The numerical experiments demonstrate that the proposed model gives about 1% to 5% more revenue than the Markov chain choice model for independent choice across different categories. The Markov chain choice model shows good flexibility in assortment problems under different business settings and tractability for parameter estimation and assortment optimization. In the future, we will work on a robust optimization formulation for this problem to address with the potentially inaccurate parameter estimation.

7 Ongoing Study: Robust Assortment Optimization

However, when we assume that the transition matrix in the secondary category N is correlated to the purchase decision in the primary category M, it leads to an explosion of parameters since we need $|M_+|\,|N_+|^2$ parameters to handle the transition of interests within the secondary category given every purchase decision in the primary category. When we do not have enough sales data of a product in the primary category, we may have estimation with large variance. Therefore, we need some special treatments to find the optimal assortment against such uncertainty. We use robust optimization with a careful construction of the uncertainty set to prevent from both out-of-sample disappointment and excessive conservation. We do not want the uncertainty set in the robust optimization contains extreme values, otherwise the solution will be very conservative. Due to the large variance of estimated $\mathbf{P}_{N|m_j}$ when the sales records of product m_j is insufficient, this estimation may be an extreme value. A possible method is to calculate a pseudo transition matrix $\mathbf{P}_N$ for the second category products. This pseudo transition matrix modified transition matrix are estimated using all the data, which reduce the variance and include more possible correlations. Then we can determine $\tilde{\mathbf{P}}_{N|m_j}$ by taking both the $\mathbf{P}_{N|m_j}$ and $\mathbf{P}_N$ into consideration. A straight forward method is to calculate a weighted sum of the transition matrix $\mathbf{P}_{N|m_j}$ and $\mathbf{P}_N$. The weights can be determined by two strategies.

7.0.1 Proportional Weights

The weights can be determined by the proportion of the purchase decision on product m_j. If we have many sales records of product m_j, then we can assign a larger weight to the transition matrix $\mathbf{P}_{N|m_j}$. A reasonable choice is to compare the number of sales records of product m_j to the average number of records of each product. Therefore, we have that

$$\tilde{\mathbf{P}}_{N|m_j} = \frac{|M|\left|T_{m_j}\right|}{|M|\left|T_{m_j}\right| + |T|}\mathbf{P}_{N|m_j} + \frac{|T|}{|M|\left|T_{m_j}\right| + |T|}\mathbf{P}_N.$$

7.0.2 Greedy Weights

We can also find the weights α that fit the sales data best. Define that $\tilde{\mathbf{P}}_{N|m_j} = \alpha_{m_j}\mathbf{P}_{N|m_j} + \left(1-\alpha_{m_j}\right)\mathbf{P}_{N|m_j}$. Recall that the purchase data is denoted by $\hat{\mathbf{Z}}_{N|m_j}$ and the probability of considering each not offered product in category N is given by $\mathbf{\Phi}_{\bar{S}_N}\left(S_M, S_N\right) = \left(\mathbf{I} - \mathbf{P}_{\bar{S}_N}\right)^{-\mathrm{T}}\mathbf{\Lambda}_{\bar{S}_N}$. We determine the weight for each product m_j by solving an optimization problem that

$$\alpha^*_{m_j} = \arg\min_{\alpha_{m_j}\in[0,1]} \|\hat{\mathbf{Z}}_{N|m_j} - \mathbf{\Lambda}_{S_N} - \tilde{\mathbf{P}}^{\mathrm{T}}_{\bar{S}_N|m_j}\mathbf{\Phi}_{\bar{S}_N}\left(S_M, S_N\right)\|_1.$$

7.0.3 Robust Formulation

Then we can formulate and solve a robust optimization problem when we want to determine the optimal assortment. Compared with the deterministic linear program, we allow the parameters varying in an uncertainty set and solve for the optimal worst-case assortment. In this problem, we allow the transition matrix $\mathbf{P}_N$ to vary. For each entry $\rho_{n_in_j}$ in the transition matrix $\mathbf{P}_N$, we construct the lower bound of the uncertainty set by the minimum of weighted parameter, which is given by $\underline{\rho}_{n_in_j} = \min\left\{\tilde{\rho}_{n_in_j|m_j} \mid m_j \in M_+\right\}$. Similarly, we construct the upper bound for $\rho_{n_in_j}$ by $\bar{\rho}_{n_in_j} = \max\left\{\tilde{\rho}_{n_in_j|m_j} \mid m_j \in M_+\right\}$. Since we do not use the deterministic transition matrix, we need to check the feasibility of each uncertain $\rho_{n_in_j}$ by adding the constraint $\sum_{n_j\in N_+}\rho_{n_in_j} = 1$ for each product n_i into the uncertainty set. Therefore, we have the uncertainty set by

$$\mathcal{U}_{\mathbf{P}_N} = \left\{\rho_{n_in_j} \in \left[\underline{\rho}_{n_in_j}, \bar{\rho}_{n_in_j}\right], \sum_{n_j\in N_+}\rho_{n_in_j} = 1 \mid n_i, n_j \in N_+\right\}.$$

Then we want to use the uncertainty set to formulate a robust optimization problem. We use the dual program of the original formulation, since it is more explainable under the setting of revenue maximization and the corresponding robust optimization problem can also be explained by maximizing the worst-case revenue. We can write the robust formulation by

$$
\begin{aligned}
\max \quad & \min_{\mathbf{P}_N \in \mathcal{U}_{\mathbf{P}_N}} \sum_{m_i \in M_+} r_{m_i} w_{m_i} + \sum_{n_i \in N_+} r_{n_i} y_{n_i} & \\
\text{s.t.} \quad & w_{m_j} + x_{m_j} - \sum_{m_i \in M_+} \rho_{m_i m_j} x_i = \lambda_{m_j} & \forall m_j \in M_+, \\
& \sum_{m_i \in M_+} \sigma_{m_i n_j} w_{m_i} + y_{n_j} + z_{n_j} - \sum_{n_i \in N_+} \rho_{n_i n_j} z_i = 0 & \forall n_j \in N_+, \\
& w_{m_j},\ x_{m_j},\ y_{n_j},\ z_{n_j} \geq 0 & \forall m_j \in M_+,\ n_j \in N_+.
\end{aligned}
\tag{6}
$$

8 References

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