EMA A plane wave has an electric field given by

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 \exp\{i(\mathbf{k}\cdot\mathbf{r} - \omega t)\}.$$

The wave propagates in an anisotropic and optically active material whose permittivity $\boldsymbol{\varepsilon}$ is a 3-by-3 hermitian matrix

$$\boldsymbol{\varepsilon} = \varepsilon_0 \begin{bmatrix} \alpha & i\beta & 0 \\ -i\beta & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}.$$

Here ε_0 is the usual permittivity of the vacuum, and α and β are real constants. The rows and columns of the matrix relate to the x, y and z directions. The electric displacement field $\mathbf{D}$ is related to $\mathbf{E}$ by the matrix product $\mathbf{D} = \boldsymbol{\varepsilon}\mathbf{E}$.

a) Use Maxwell's equations to obtain a matrix equation obeyed by the vector $\mathbf{E}_0$ whose solution will allow you to find the possible polarization directions and values of ω for each wave vector $\mathbf{k}$. [Hint: Note that $\nabla\cdot\mathbf{E}$ is not necessarily zero. Also you may find the identity

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

to be of use.]

b) Find the eigenvalues and eigenvectors of the matrix $\boldsymbol{\varepsilon}$.

c) Consider the propagation of plane waves in the z direction. Use your results from parts (a) and (b) to find the two allowed polarization vectors and dispersion relations (*i.e.* how does ω depend on k_z) for these waves.

d) The matrix $\boldsymbol{\varepsilon}$ has *three* eigenvectors. Explain why your answer to part (a) shows that only *two* of them are valid polarization vectors?

e) In the plane $z = 0$ the electric field for a wave polarized in the x direction and propagating in the z direction is given by

$$\mathbf{E}(t) = |\mathbf{E}_0|\mathbf{e}_x \exp\{-i\omega t\},$$

where $\mathbf{e}_x$ is the unit vector in the x direction. Find the angle through which the polarization has rotated after the wave has propagated a distance d.