BQM. In parts (a) and (b) of this problem you can ignore any spin possessed by the quantum particle.

a) A quantum particle moving in the $z = 0$ plane has a wavefunction that in r, θ, polar coordinates is given by

$$\chi(r, \theta) \equiv \langle r, \theta|\chi\rangle = a(r)\exp\{in\theta\}, \quad n \text{ an integer.}$$

Let $\hat{L}_z$ be the angular momentum operator that acts on states $|\psi\rangle$ as

$$\langle r, \theta|\hat{L}_z|\psi\rangle = -i\hbar\frac{\partial}{\partial\theta}\langle r, \theta|\psi\rangle,$$

Show that $|\chi\rangle$ is an eigenstate of the operator $\hat{R}(\varphi) = \exp\{-i\varphi\hat{L}_z/\hbar\}$, where φ is a real parameter. What is the eigenvalue?

b) Another state $|\Psi\rangle$ has a wavefunction that can be expanded as

$$\langle r, \theta|\Psi\rangle = \sum_{n=-\infty}^{\infty} a_n(r)\exp\{in\theta\}.$$

By using your result from part (a) compute $\langle r, \theta|\hat{R}(\varphi)|\Psi\rangle$ and hence show that $\hat{R}(\varphi)|\Psi\rangle$ can be interpreted as the state $|\Psi\rangle$ after it has been rotated through some angle about the origin. Through what angle and in which sense (clockwise or anticlockwise) is the rotation if $\varphi = \pi/2$?

In the remaining parts of the problem we consider the effect of rotations on the spin part of the wavefunction of a spin-1/2 particle. Here

$$\hat{\mathbb{I}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

c) By expanding the exponential, show that $\exp\{-i\varphi\hat{S}_x/\hbar\} = A\hat{\mathbb{I}} + B\hat{S}_x$ where A and B are functions of φ that you should find.

d) Let

$$|\uparrow\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\Phi\rangle \equiv \exp\{-i(\pi/2)\hat{S}_x/\hbar\}|\uparrow\rangle$$

Compute $\langle\Phi|\hat{S}_x|\Phi\rangle$, $\langle\Phi|\hat{S}_y|\Phi\rangle$, $\langle\Phi|\hat{S}_z|\Phi\rangle$. Demonstrate that your results are compatible with interpreting the action of $\exp\{-i(\pi/2)\hat{S}_x/\hbar\}$ as a rotation about the x axis.