AQM. In the absence of any spin-orbit coupling the bound eigenstates of the single-electron hydrogen atom are denoted by $|n, l, m_l, m_s\rangle$. They have energy $E_n = -R/n^2$ where R is the Rydberg constant. You will consider the effect of including a spin-orbit interaction

$$\hat{H}_{\mathrm{spin-orbit}} = \beta\,\hat{\mathbf{L}} \cdot \hat{\mathbf{S}},$$

which we will treat as a perturbation. Here β is a real parameter, $\hat{\mathbf{L}}$ is the operator corresponding to orbital angular momentum of the electron, and $\hat{\mathbf{S}}$ is its spin operator.

a) Before the spin-orbit interaction is switched on, what is the degeneracy of the $n = 2$ energy level? List the possible states using the notation $|n, l, m_l, m_s\rangle$.

b) Once the spin-orbit interaction is switched on, the degeneracy of the E_n level is partially lifted and l, m_l, and m_s are no longer "good" quantum numbers (*i.e.* they are no longer the eigenvalues of operators that commute with the Hamiltonian). What is a new set of good quantum numbers?

c) Consider the perturbed eigenstates that arise from the $n = 2$, $l = 1$ states. Compute, to first order in β, the shift in the energies from $E_{n=2}$. How many distinct energies are there after the perturbation, and what is the degeneracy of each?

d) For the energy level with the highest degeneracy, write down all the perturbed eigenstates as explicit linear combinations of the original $n = 2$, $l = 1$ basis states $|2, 1, m_l, m_s\rangle$.

Hint:

$$\hat{\mathbf{J}}^2 \equiv (\hat{\mathbf{L}} + \hat{\mathbf{S}})^2 = \hat{\mathbf{L}}^2 + \hat{\mathbf{S}}^2 + 2\,\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}.$$

Some possibly useful Clebsch-Gordan coefficients $\langle j_1, m_1; j_2, m_2 | j_{\mathrm{tot}}, m_{\mathrm{tot}}\rangle$:

$$\langle 1, m_l; \tfrac{1}{2}, \pm\tfrac{1}{2} | \tfrac{3}{2}, m_{\mathrm{tot}}\rangle = \sqrt{\frac{\frac{3}{2} \pm m_{\mathrm{tot}}}{3}}; \quad \langle 1, m_l; \tfrac{1}{2}, \pm\tfrac{1}{2} | \tfrac{1}{2}, m_{\mathrm{tot}}\rangle = \pm\sqrt{\frac{\frac{3}{2} \mp m_{\mathrm{tot}}}{3}}.$$