QMB The Hamiltonian of an electron at rest in magnetic field $\mathbf{B}$ is given by

$$H = \frac{e\hbar}{2mc}\boldsymbol{\sigma}\cdot\mathbf{B}.$$

Here $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, where σ_i are the Pauli matrices. Suppose that $\mathbf{B} = B_z\mathbf{e}_z + B_0\mathbf{e}_x$, where $B_z > 0$ and $B_0 > 0$ are constant. Let θ be the angle between $\mathbf{B}$ and the z-axis. The Hamiltonian is therefore

$$H = \frac{\hbar}{2}\begin{bmatrix}\omega_z & \omega_0\\ \omega_0 & -\omega_z\end{bmatrix} = \frac{\hbar\sqrt{\omega_z^2+\omega_0^2}}{2}\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}.$$

where $\omega_z = eB_z/mc$ and $\omega_0 = eB_0/mc$.

a) Compute the eigenvalues and normalized eigenvectors of H. (You may find the following half-angle identities useful: $\sin(\theta/2) = \sqrt{(1-\cos\theta)/2}$ and $\cos(\theta/2) = \sqrt{(1+\cos\theta)/2}$).

b) Let $|+\rangle$ and $|-\rangle$ denote the spin-up and down eigenstates of σ_z. Express $|+\rangle$ in terms of the eigenstates of H from part (a).

c) Suppose you find the electron in the $|+\rangle$ state at time $t = 0$. Compute the probability that it will be in the $|-\rangle$ state at time $t > 0$. Express your answer in terms of ω_0 and ω_z.

Suppose now that the magnetic field is replaced by a time dependent field $\mathbf{B}(t) = B_z\mathbf{e}_z + B_0\cos(\omega t)\mathbf{e}_x + B_0\sin(\omega t)\mathbf{e}_y$, where $B_z, B_0, \omega > 0$ are constants.

d) Let $|\psi(t)\rangle$ denote the state vector for this new system. Derive the Schrödinger equation satisfied by the transformed state $|\tilde{\psi}(t)\rangle = R(t)|\psi(t)\rangle$, where

$$R(t) = \begin{bmatrix}e^{i\omega t/2} & 0\\ 0 & e^{-i\omega t/2}\end{bmatrix}.$$

Note that $|\tilde{\psi}(t)\rangle$ is the state vector in the frame that is co-rotating with the time-dependent $\mathbf{B}$ field.

e) If the electron is in the $|+\rangle$ state at time $t = 0$, what is the probability that it will be in the $|-\rangle$ state at time $t > 0$. Express your answer in terms of $\omega_z, \omega_0, \omega$.