Q1 Consider the $n = 1$ and $n = 2$ states of an electron in a hydrogen atom. The energy eigenfunctions $\psi_{nlm}(r, \theta, \phi)$ are

$$\begin{aligned}
\psi_{100}(r, \theta, \phi) &= \frac{2}{a^{3/2}} e^{-r/a} Y_0^0(\theta, \phi), \\
\psi_{200}(r, \theta, \phi) &= \frac{1}{\sqrt{2} a^{3/2}} \left(1 - \frac{r}{2a}\right) e^{-r/2a} Y_0^0(\theta, \phi), \\
\psi_{21m}(r, \theta, \phi) &= \frac{1}{\sqrt{24} a^{3/2}} \left(\frac{r}{a}\right) e^{-r/2a} Y_1^m(\theta, \phi), \quad (m = 0, \pm 1).
\end{aligned}$$

Here a is the Bohr radius. Expressions for the spherical harmonics $Y_l^m(\theta, \phi)$ may be found in the formulæ pages at the beginning of the exam.

The hydrogen atom is now immersed in a uniform electric field that adds a perturbation

$$H' = eEz = eEr\cos\theta$$

to the original Hamiltonian.

a) Use first-order perturbation theory to show that the shift of the ground state energy due to H' is zero. You may do this by either explicit calculation, or by applying selection rules based on angular-momentum addition rules and/or parity.

b) The four $n = 2$ states have the same energy, and so degenerate perturbation theory must be used. By the application of selection rules and symmetries (or by explicit calculation) show that of the 16 matrix elements $\langle 2, l, m|H'|2, l', m'\rangle$, only one pair, consisting of a matrix element and its complex conjugate, can be non-zero. Evaluate these two non-zero matrix elements and express them in terms of a, e and E.

c) Use the 4-by-4 matrix $\langle 2, l, m|H'|2, l', m'\rangle$ to compute the energy shifts of the $n = 2$ energy levels.

Useful Integral:

$$\int_0^\infty dr\, r^n e^{-r/b} = n! b^{n+1}.$$