Q3 A one-dimensional harmonic oscillator has the quantum Hamiltonian

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2, \quad [\hat{x}, \hat{p}] = i\hbar.$$

Recall that the normalized energy eigenstates are given by $|n\rangle = (\hat{a}^\dagger)^n|0\rangle/\sqrt{n!}$ where

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - i\sqrt{\frac{1}{2\hbar m\omega}}\hat{p}, \quad \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\sqrt{\frac{1}{2\hbar m\omega}}\hat{p},$$

and $|0\rangle$ is the ground state, which obeys $\hat{a}|0\rangle = 0$.

a) Compute the commutators $[\hat{a}, \hat{a}^\dagger]$, $[\hat{H}, \hat{a}]$ and $[\hat{H}, \hat{a}^\dagger]$, and hence evaluate the time-dependent operator

$$\hat{a}(t) \stackrel{\text{def}}{=} e^{i\hat{H}t/\hbar}\,\hat{a}\,e^{-i\hat{H}t/\hbar}$$

in terms of $\hat{a}$, m, ω and $\hbar$.

b) Given a complex parameter α, we define the corresponding *coherent state* $|\alpha\rangle$ to be

$$|\alpha\rangle = e^{|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle.$$

Show that $|\alpha\rangle$ is a normalized eigenstate of $\hat{a}$. What is the associated eigenvalue?

c) If the system is in the coherent state $|\alpha\rangle$, what is the probability that the system is in the energy eigenstate $|n\rangle$?

d) If the system is the state $|\alpha\rangle$ at time $t = 0$, show that at time t it is in another coherent state $|\beta\rangle$. How is the parameter β related to α?

e) Compute $X = \langle\alpha|\hat{x}|\alpha\rangle$ and $P = \langle\alpha|\hat{p}|\alpha\rangle$. Then, using your results from parts (a),(b) and (d), evaluate $X(t)$, $P(t)$. Draw a figure showing the trajectory of the point $(X(t), P(t))$ in the X, P plane, and relate your plot to some aspects of the classical harmonic oscillator.