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Consider an isotropic two-dimensional harmonic oscillator of unit mass. The coordinates in the plane of oscillation are x, y. The quantum Hamiltonian is given by

$$H = -\frac{\hbar^2}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{\omega^2}{2}(x^2 + y^2),$$

where ω is the frequency of the oscillator.

(a) As a trial wave function, use

$$\Psi_0 = N \exp(-\frac{\alpha}{2}(x^2 + y^2)),$$

where N is a normalization constant and α is a constant to be determined. Using the Schrodinger equation, demand that Ψ_0 is an energy eigenstate, and find the value of α and the energy E_0 of the state. Leave N undetermined. Give a brief argument as to why Ψ_0 is or is not the ground state of the system.

(b) It is useful in this problem to define $x_\pm = x \pm iy, p_\pm = p_x \pm ip_y$. Using the basic commutation rules between coordinates and momenta, determine the four commutators: $[p_a, x_b]$ for $a = \pm, b = \pm$. Write the Hamiltonian in terms of $p_\pm$ and $x_\pm$.

(c) Define the operator $B^\dagger$ by $B^\dagger = \omega x_+ - ip_+$. Use the results of (b) to show that the commutator $[H, B^\dagger]$ is of the form

$$[H, B^\dagger] = \beta B^\dagger,$$

where β is a constant. Determine the value of β.

(d) Consider the state $B^\dagger\Psi_0$. Using the results of (c), show that this state is an eigenstate of H, and determine its energy. Give the formula for $B^\dagger\Psi_0$ explicitly in terms of x, y, up to an undetermined normalization constant. Will applying $B^\dagger$ repeatedly to Ψ_0 lead to additonal eigenstates? Explain your answer.

(e) The isotropy of the oscillator leads to a conserved angular momentum operator L given by

$$L = \frac{i}{2}(x_+p_- - x_-p_+).$$

Show that L is conserved. The states Ψ_0 and $B^\dagger\Psi_0$ are eigenstates of L. Determine their eigenvalues.