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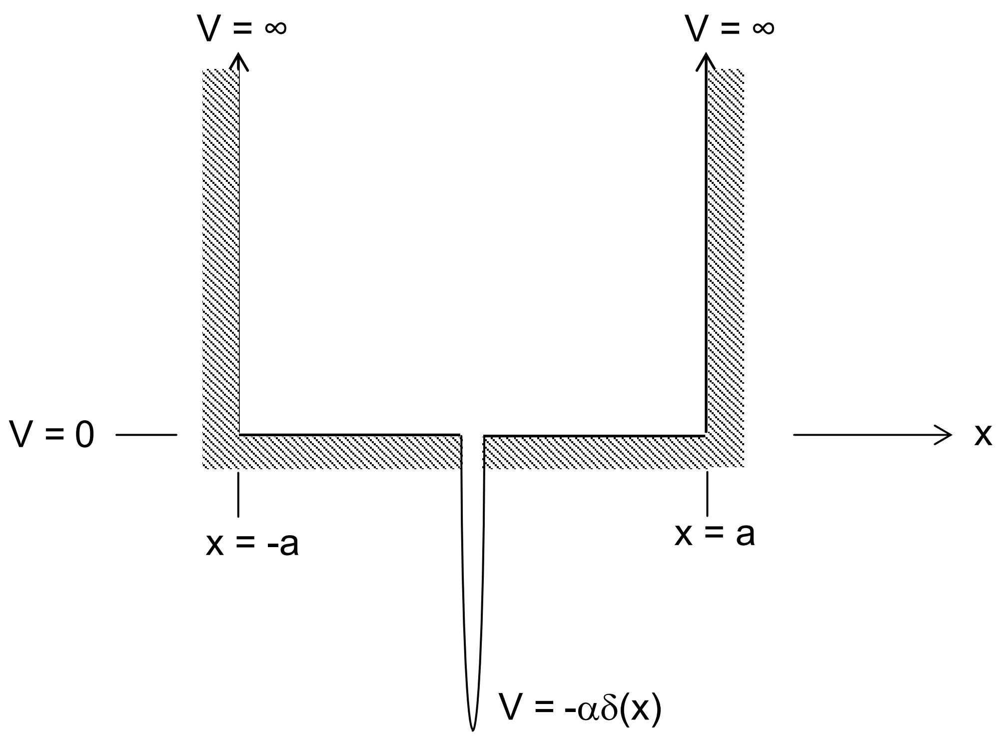
A particle of mass m moves in a one-dimensional potential well having both infinitely high potential walls at $x = \pm a$, and an attractive one-dimensional δ-function potential well of strength α located at $x = 0$:

$$V(x) = -\alpha\delta(x) \qquad |x| < a$$
$$V(x) = \infty \qquad |x| \geq a$$



(a) Show by integrating the Schrödinger equation across the δ-function that the wavefunction for this potential exhibits a discontinuous change of slope across the δ-function given by:

$$d\psi / dx\big|_{0+} - d\psi / dx\big|_{0-} = -\left(2m\alpha / \hbar^2\right)\psi(0)$$

(b) Assuming that α is small, qualitatively sketch the wavefunctions associated with (i) the ground state, (ii) the 1st excited state, and (iii) the 2nd excited state of the particle confined in the potential above.

(c) Assume that α is small, so that the δ-function potential can be treated as a small perturbation on the infinite square well potential ($\alpha = 0$). <u>For the lowest three energy levels</u>, use first-order perturbation theory to estimate the energy difference $\Delta E_n^{(1)}$ between the eigenvalues of the full potential shown above ($\alpha \neq 0$), and those of the infinite square well potential ($\alpha = 0$).

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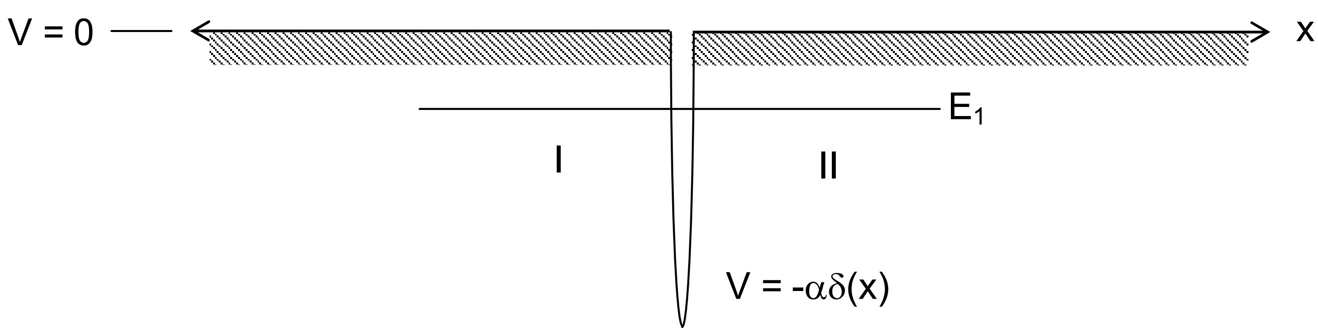
(d) If the infinite potential walls are moved to $x = \pm\infty$ (as shown below), for finite α, one can show that there will be one bound state of the δ-function potential with an energy E_1. Sketch this bound state wavefunction of the δ-function potential.



(e) Use the variational trial wavefunction $\psi_{tr} = Ae^{-bx^2}$ to obtain an upper-bound on the bound state energy of the δ-function potential E_1.

The following integrals may be useful:

$$\int_{-\infty}^{\infty} x^{2n} e^{-ax^2} dx = \frac{1\cdot 3\cdot 5\cdots}{2^n a^n}\sqrt{\frac{\pi}{a}} \qquad \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \; .$$