A particle moves in a three-dimensional harmonic oscillator potential

$$V(\vec{r}) = \frac{m\omega^2}{2}\left(x^2 + y^2 + z^2\right)$$

(a) Define raising and lowering operators by

$$a_x = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega}p\right),\ a_x^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{i}{m\omega}p\right),\ \text{etc.}$$

and similarly for the y and z directions. Write the position operator x in terms of them. In terms of the raising and lowering operators, write down the form of the ground state and excited states of the system.

(b) A weak perturbation of the form

$$\delta V = Uxyz + \frac{U^2}{\hbar\omega}x^2y^2z^2$$

is applied. Show that, to first order in U, the correction to the ground state energy is zero.

(c) Work out the ground state energy up to and including $O(U^2)$. Be careful to include all contributions.