The helium atom consists of a nucleus with charge +2e and no spin, and two electrons, each with charge –e and spin ½ħ. In this problem, treat the nucleus as a fixed (immovable) source of electric field. Also, ignore the spin-orbit interaction and other fine structure and hyperfine effects.

(a) Write the Schrödinger equation that describes this problem.

In the following two parts, ignore the interaction between the two electrons.

(b) Write a normalized 2-electron state, including the spin and spatial degrees of freedom, that has the minimum (ground state) energy. How many states are degenerate with it?

(c) What are the energies, in electron-Volts (eV), of the 2-electron ground state, E_g, and first excited state, E_2?

Now, take into account the Coulomb interaction between the electrons.

(d) The integrals needed to calculate the electron-electron interaction energy, E_{ee}, are too tedious for a qual problem. **Do not evaluate any integrals.** Inspection of the integrals will reveal that the dependence of E_{ee} on the nuclear charge, Z, is $E_{ee} \propto Z^g$ (assuming that the wave functions of the two electrons are not distorted by the e-e interaction). What is g ?

(e) Assume that you've done the integral in part **(d)** for $Z = 2$, obtaining $E_{\mathrm{ee}} = b|E_{\mathrm{g}}|$ (b is a positive number). Set up and solve a variational calculation of the 2-electron ground state energy. The variation is to let the wave function of each of the two electrons be that of an atom of some other (not necessarily integer) Z. Call it Z'. The two electrons are described by the same Z'. Determine the value of Z' that minimizes the helium ground state energy.

NOTE: The ground state wave function of an electron in the hydrogen atom is:

$$y_{100}(r) = \left(\frac{1}{pa^3}\right)^{1/2} e^{-r/a},$$

where $a = \hbar^2/me^2$, and the ground state energy is $E_1 = -me^4/2\hbar^2 = -13.6$ eV.