Consider an electron in a spherically symmetric potential . The spatial wave function of the electron in the eigenstate with energy E_1 of the corresponding Hamiltonian has the form

$$\psi_1(\vec{r}) = xf(r),$$

where x is the Cartesian component of the position vector $\vec{r}$, and r denotes the magnitude of $\vec{r}$. $f(r)$ is a spherically symmetric function. ψ_1 satisfies the normalization condition

$$\int d^3r |\psi_1(\vec{r})|^2 = 1.$$

(a) Is $\psi_1(\vec{r})$ an eigenfunction of the orbital angular momentum operator $L_z = -i\hbar\frac{\partial}{\partial\phi}$ (where ϕ is the azimuthal angle)? Briefly explain your answer.
(b) Write down all the linearly-independent wave functions associated with the same energy E_1 that are related by rotational symmetry.
(c) A spin-orbit interaction of the form

$$V_{so} = U(r)\vec{S}\cdot\vec{L}/\hbar^2$$

perturbs the system. Here $U(r)$ is a spherical potential, $\vec{L}$ denotes the orbital angular momentum operator, and $\vec{S}$ denotes the spin angular momentum operator.
Find the first order energy splitting of the E_1 level due to the spin-orbit interaction in terms of Δ, where

$$\Delta \equiv \int d^3r |\psi_1(\vec{r})|^2 U(r).$$

(d) Initially an electron is in the state $|\psi_1\rangle$ with spin up, where the quantization axis is along z. The spin-orbit interaction remains on. Rewrite the initial state in terms of linear combinations of $|j,m\rangle$ (the eigen-states of J^2 and J_z, where J is the total spin angular momentum) in the form $\sum_{jm} C(j,m)|j,m\rangle$ and find the coefficients $C(j,m)$. You may utilize the Clebsch-Gordan coefficient table provided below. Find the probability that the electron remains in the state $|\psi_1\rangle$ with spin up after a time t.

Table 1: Clebsch-Gordan coefficients

1 × 1/2 ‖	(3/2,3/2)	(3/2,1/2)	(3/2,-1/2)	(3/2,-3/2)	(1/2,1/2)	(1/2,-1/2)
(1,1/2)	1	0	0	0	0	0
(1,-1/2)	0	$\sqrt{1/3}$	0	0	$\sqrt{2/3}$	0
(0,1/2)	0	$\sqrt{2/3}$	0	0	$-\sqrt{1/3}$	0
(0,-1/2)	0	0	$\sqrt{2/3}$	0	0	$\sqrt{1/3}$
(-1,1/2)	0	0	$\sqrt{1/3}$	0	0	$-\sqrt{2/3}$
(-1,-1/2)	0	0	0	1	0	0