This problem concerns a quantum mechanical system of 2 identical spin 1/2 fermions in parts (a), (b), and (c). Parts (d), (e), and (f) concern a system of three identical spin 1/2 fermions. The two fermion system consists of fermions 1 and 2, while the three fermion system consists of fermions 1, 2 and 3. In this notation, for the two fermion system if 1 had spin up and 2 has spin down, the spin state would be written as $|+->$.

(a) For the two fermion system, the values of the total spin quantum number are $S = 0, 1$. Construct the three normalized spin states of $S = 1$, labelled by the values of $S_z = 0, \pm 1$.

(b) Construct the normalized state of $S = S_z = 0$.

(c) Suppose now the system is one dimensional in space. The two fermions can be in either of the two (normalized, orthogonal) spatial states $\psi_a(x)$ and $\psi_b(x)$. The two fermion spatial wave function is

$$\frac{1}{\sqrt{2}}(\psi_a(x_1)\psi_b(x_2) - \psi_b(x_1)\psi_a(x_2)),$$

where x_1 and x_2 are the spatial coordinates of fermions 1 and 2, respectively. Write down a valid total wave function (spin and space) which satisfies the Pauli principle, and has $S_z = 0$.

The rest of the problem involves a three spin system

(d) Now turn to a system of 3 spin 1/2 particles, 1, 2, and 3. Write down the four normalized states of total spin $S = 3/2$, corresponding to $S_z = \pm 3/2, \pm 1/2$.

(e) There are two states of total spin $S = 1/2$. Find one of them with $S_z = 1/2$ by forming a state which is symmetric in the interchange of spins 1 and 2, and is orthogonal to the state with $S = 3/2, S_z = 1/2$.

(f) Now consider the spatial states of the three particles in one dimension and assume the particles can be in the normalized and orthogonal states $\psi_a(x)$, $\psi_b(x)$, or $\psi_c(x)$). The system is known to have $S = 3/2, S_z = 1/2$. Which of the following spacial wave functions for the system of three spins will allow the Pauli Principle to be satisfied?

$$\Psi = \frac{1}{\sqrt{6}}(\psi_a(x_1)\psi_b(x_2)\psi_c(x_3) + \psi_a(x_1)\psi_b(x_3)\psi_c(x_2) + \psi_a(x_2)\psi_b(x_3)\psi_c(x_1)$$

$$+\psi_a(x_2)\psi_b(x_1)\psi_c(x_3) + \psi_a(x_3)\psi_b(x_1)\psi_c(x_2) + \psi_a(x_3)\psi_b(x_2)\psi_c(x_1)),$$

or

$$\Psi' = \frac{1}{\sqrt{6}}(\psi_a(x_1)\psi_b(x_2)\psi_c(x_3) - \psi_a(x_1)\psi_b(x_3)\psi_c(x_2) + \psi_a(x_2)\psi_b(x_3)\psi_c(x_1)$$

$$-\psi_a(x_2)\psi_b(x_1)\psi_c(x_3) + \psi_a(x_3)\psi_b(x_1)\psi_c(x_2) - \psi_a(x_3)\psi_b(x_2)\psi_c(x_1)),$$

Choose either Ψ or Ψ' and briefly explain your reasoning.